This problem set serves as general revision on the Baire Lemma and its applications.

3.1. Quick warm-up: true or false?

Decide whether the following statements are true or false. If true, think of a quick proof. If false, find a simple counterexample. *(self-check: not to be handed in.)*

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If there exists $f\colon [0,1] \to \mathbb{R}$ such that $\forall x \in [0,1] : \lim_{n\to\infty} f_n(x) = f(x)$ then $f \in C^0([0,1])$.

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If $\forall x \in [0,1]$ $\exists\, C(x) : \sup_{n\in\mathbb{N}} |f_n(x)| \leq C(x)$ then $\sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |f_n(x)| < \infty$.

(c) The function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $d(x,y) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, is a distance.

(d) There exists $A \subset \mathbb{R}$ such that both A and its complement $A^{\complement}$ are dense in $\mathbb{R}$.

(e) $\big(C^1([-1,1]), \|\cdot\|_{C^0}\big)$ is a Banach space, i. e. a complete normed space.

(f) The complement of a 2^{nd} category set is a 1^{st} category set.

(g) A nowhere dense set is meagre.

(h) A meagre set is nowhere dense.

(i) Let U be the set of fattened rationals in $\mathbb{R}$, namely

$$U := \bigcap_{j=1}^{\infty} U_j, \qquad\qquad U_j := \bigcup_{k=1}^{\infty} \big]q_k - 2^{-(j+k+1)}, q_k + 2^{-(j+k+1)}\big[,$$

where $(q_n)_{n\in\mathbb{N}}$ is a counting of $\mathbb{Q}$. Then $U = \mathbb{Q}$.

3.2. An application of Baire

Let $f \in C^0([0,\infty[)$ be a continuous function satisfying

$$\forall t \in [0,\infty[: \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Hint. Apply the Baire Lemma as in the proof of the uniform boundedness principle.

3.3. Compactly supported sequences and their ℓ^∞-completion

Definition. We denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N} \; \forall n \geq N : \; x_n = 0\}$$

and the space of sequences converging to zero by

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

(a) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is the completion of this space?

(b) Prove the strict inclusion

$$\bigcup_{p=1}^{\infty} \ell^p \subsetneq c_0.$$

3.4. Discontinuous at irrationals

Definition. Let (X,d) be a metric space. A subset $A \subset X$ is called $\mathfrak{G}_\delta$-set of X (say: "G-delta-set") if there exist open sets $O_n \subset X$ for $n \in \mathbb{N}$ such that $A = \bigcap_{n\in\mathbb{N}} O_n$.

(a) Let $f\colon X \to \mathbb{R}$ be any function. Show that the set of points $x \in X$ at which f is continuous is a $\mathfrak{G}_\delta$-set.

(b) Suppose that (X,d) is complete. Suppose also that $A \subset X$ and $A^\complement = X \setminus A$ are both dense in X. Show that at most one of A and $A^\complement$ is a $\mathfrak{G}_\delta$-set.

(c) Show that there is no function $f\colon [0,1] \to \mathbb{R}$ that is continuous at all rationals and discontinuous at all irrationals.

Hint. For (c), apply (a) and (b) with $X = [0,1]$ and d the Euclidean distance.