4.1. Algebraic basis

Definition. Let X be a vector space. An *algebraic basis* for X is a subset $E \subset X$ such that every $x \in X$ is uniquely given as *finite* linear combination of elements in E.

(a) Let $(X, \|\cdot\|)$ be a complete normed space. Show that any algebraic basis for X is either finite or uncountable.

Hint. Assume that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$ and derive a contradiction to the Baire Lemma by considering the sets $A_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

(b) Find an example of a normed space whose algebraic basis is countably infinite.

4.2. Closed subspaces

Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N} : \ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N} : \ x_{2n-1} = nx_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. Prove that if any sequence $(x^{(k)})_{k\in\mathbb{N}}$ of elements $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in \ell^1$ converges to $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 for $k \to \infty$, then each entry $x_n^{(k)}$ converges in $\mathbb{R}$ to x_n for $k \to \infty$. For the second claim, show $c_c \subset U \oplus V$. (Recall c_c from problem 3.3 or 4.6.)

4.3. Normal convergence

Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|)$ is a Banach space.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{k=1}^{\infty}\|x_n\| < \infty$ the limit $\lim_{N\to\infty}\sum_{n=1}^{N} x_n$ exists.

Hint. A Cauchy sequence converges if and only if it has a convergent subsequence.

4.4. Subsets with compact boundary

Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space an let $Z \subset X$ be a bounded subset with compact boundary. Prove that Z has empty interior: $Z^\circ = \emptyset$.

Hint. Assume that $Z^\circ \neq \emptyset$. Find a continuous functional that projects the boundary ∂Z to the boundary of a ball inside Z. This will contradict the fact that the unit sphere in an infinite-dimensional normed space is non-compact.

4.5. Approaching the sign function

We consider the space $X = C^0([-1,1],\mathbb{R})$ with its usual norm $\|\cdot\|_{C^0([-1,1])}$ and define

$$\begin{aligned} \varphi\colon X &\to \mathbb{R} \\ f &\mapsto \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t. \end{aligned}$$

(a) Show that $\varphi \in L(X,\mathbb{R})$ with $\|\varphi\|_{L(X,\mathbb{R})} \leq 2$.

(b) Find a sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $\|f_n\|_{C^0([-1,1])} = 1$ for every $n \in \mathbb{N}$ and such that $\varphi(f_n) \to 2$ as $n \to \infty$. This in fact implies $\|\varphi\|_{L(X,\mathbb{R})} = 2$.

(c) Prove that there does not exist $f \in X$ with $\|f\|_{C^0([-1,1])} = 1$ and $|\varphi(f)| = 2$.

4.6. Unbounded map and approximations

As in problem 3.3, we denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N: \ x_n = 0\}$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$. Consider the map

$$\begin{aligned} T\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (nx_n)_{n\in\mathbb{N}} \end{aligned}$$

(a) Show that T is not continuous.

(b) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c: \quad T_m x \xrightarrow{m\to\infty} Tx.$$