5.1. Operator norm

(a) Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map from $\mathbb{R}^n$ to itself. Show that the squared operator norm $\|A\|^2$ equals the largest eigenvalue of $A^{\intercal}A$.

(b) Let $A \in L(\mathbb{R}^{2017}, \mathbb{R}^{2017})$ be symmetric such that there exists a basis $\mathcal{B}$ of $\mathbb{R}^{2017}$ diagonalising A with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2017}\} = \{1, 2, \ldots, 2017\}$ each with multiplicity one.

Let $B \in L(\mathbb{R}^{2017}, \mathbb{R}^{2017})$ be symmetric such that there exists a basis $\mathcal{B}'$ not necessarily equal to $\mathcal{B}$ of $\mathbb{R}^{2017}$ diagonalising B with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2017}\} = \{1, 2, \ldots, 2017\}$ each with multiplicity one.

Prove that the operator norm of the composition BA can be estimated by

$$\|BA\| < 4\,410\,000.$$

5.2. Volterra equation

Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Show that for every $g \in C^0([0,1])$ there exists a unique $f \in C^0([0,1])$ satisfying

$$\forall t \in [0,1]: \quad f(t) + \int_0^t k(t,s) f(s)\,\mathrm{d}s = g(t).$$

Hint. Choose a space $(X, \|\cdot\|_X)$ and show that the operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s) f(s)\,\mathrm{d}s$$

has spectral radius $r_T = 0$. Then apply Satz 2.2.7.

5.3. Right shift operator

The right shift map on the space ℓ^2 is given by

$$\begin{aligned} S\colon \ell^2 &\to \ell^2 \\ (x_1, x_2, \ldots) &\mapsto (0, x_1, x_2, \ldots). \end{aligned}$$

(a) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(b) Compute the eigenvalues and the spectral radius of S.

(c) Show that S has a left inverse in the sense that there exists an operator $T\colon \ell^2 \to \ell^2$ with $T \circ S = \mathrm{id}\colon \ell^2 \to \ell^2$. Check that $S \circ T \neq \mathrm{id}$.

5.4. Closed subspaces

Let $(X, \|\cdot\|_X)$ be a normed space an let $U, V \subset X$ be subspaces. Prove the following.

(a) If U is finite dimensional and V closed, then $U + V$ is a closed subspace of X.

(b) If V is closed with finite codimension, i. e. $\dim(X/V) < \infty$, then $U + V$ is closed.

Hint. Is the canonical quotient map $\pi\colon X \to X/V$ continuous? What is $\pi^{-1}(\pi(U))$?

5.5. Vanishing boundary values

Let $X = C^0([0,1])$ and $U = C^0_0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}$.

(a) Show that U is a closed subspace of X endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$.

(b) Compute the dimension of the quotient space X/U and find a basis for X/U.

5.6. Topological complement

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space. A subspace $U \subset X$ is called *topologically complemented* if there is a subspace $V \subset X$ such that the linear map I given by

$$\begin{aligned} I\colon (U \times V, \|\cdot\|_{U\times V}) &\to (X, \|\cdot\|_X), \qquad\qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X, \\ (u,v) &\mapsto u + v \end{aligned}$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(a) Prove that $U \subset X$ is topologically complemented if and only if there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(b) Show that a topologically complemented subspace must be closed.

5.7. Continuity of bilinear maps

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(a) Show that B is continuous if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y: \quad \|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y. \tag{$\dagger$}$$

(b) Assume that $(X, \|\cdot\|_X)$ is complete. Assume further that the maps

$$\begin{aligned} X &\to Z & \qquad\qquad Y &\to Z \\ x &\mapsto B(x, y') & y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then, $(\dagger)$ holds.

Hint. Apply the Theorem of Banach-Steinhaus to a suitable map but recall that it requires completeness of the domain.