6.1. Graph of bounded functions

Let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded and let $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\} \subset \mathbb{R}^2$ be its graph. Show that f is continuous if and only if Γ is closed.

Is the same statement true if f is *not* assumed to be bounded?

6.2. An implication of Hellinger-Töplitz (coercive operators)

Let $(H, (\cdot, \cdot))$ be a Hilbert space and let $A\colon H \to H$ be a symmetric linear operator such that

$$\exists \lambda > 0 \quad \forall x \in H : \quad (Ax, x) \geq \lambda \|x\|^2.$$

(Any linear operator satisfying such an inequality is called *coercive.*) Show that A is an isomorphism of normed spaces and $\|A^{-1}\| \leq \lambda^{-1}$.

Hint. To prove surjectivity, i. e. $W_A := A(H) = H$, consider an element $x \in W_A^{\perp}$ and recall that $(W_A^{\perp})^{\perp} = \overline{W_A}$. (*Achtung*, do not forget the closure.)

6.3. Derivative operator

Let $X = L^2([0, 1])$. On $D_A := C_c^{\infty}(]0, 1[) \subset X$ we define the derivative operator

$$\begin{aligned} A\colon D_A &\to X \\ f &\mapsto f'. \end{aligned}$$

Recall that A is closable. Show that the domain $D_{\overline{A}}$ of its closure is contained in

$$\{f \in C^0([0, 1]) \mid f(0) = 0 = f(1)\}.$$

Hint. Given $f \in D_{\overline{A}}$ consider a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A which converges to f in X. *Achtung*, L^2-convergence does *not* imply pointwise convergence: You can't evaluate f at points. Instead, compare $f_n(t)$ to $g(t) := \int_0^t \overline{A} f \, dx$.

6.4. Closed range

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a linear operator with closed graph. Show that the following statements are equivalent.

(i) A is injective and its range $W_A := A(D_A)$ is closed in $(Y, \|\cdot\|_Y)$.

(ii) $\exists C > 0 \quad \forall x \in D_A : \quad \|x\|_X \leq C\|Ax\|_Y$

Hint. One implication follows from the Inverse Mapping Theorem.

6.5. Graph norm

(a) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $A\colon D_A \subset X \to Y$ be a linear operator with graph $\Gamma_A \subset X \times Y$. Then, $\|x\|_{\Gamma_A} := \|x\|_X + \|Ax\|_Y$ defined on D_A is called the *graph norm*.

Show that if A has closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space.

(b) Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be Banach spaces and let

$$\begin{aligned} &T_1\colon D_1 \subset X_0 \to X_1, \\ &T_2\colon D_2 \subset X_0 \to X_2 \end{aligned}$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that

$$\exists C > 0 \quad \forall x \in D_1 : \quad \|T_2 x\|_{X_2} \leq C\Big(\|T_1 x\|_{X_1} + \|x\|_{X_0}\Big).$$

6.6. Closed sum

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let

$$\begin{aligned} &A\colon D_A \subset X \to Y, \\ &B\colon D_B \subset X \to Y \end{aligned}$$

be linear operators with $D_A \subset D_B$. Under the assumption that there exist constants $0 \leq a < 1$ and $b \geq 0$ such that

$$\forall x \in D_A\colon \quad \|Bx\|_Y \leq a\|Ax\|_Y + b\|x\|_X,$$

show that if A has closed graph, then $(A + B)\colon D_A \to Y$ has closed graph.

Hint. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in D_A, prove the estimate

$$(1 - a)\|A(x_n - x_m)\| \leq \|(A + B)(x_n - x_m)\| + b\|x_n - x_m\|.$$

6.7. Closable inverse

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A}$ is injective. Show that then, the inverse operator A^{-1} is closable and $\overline{A^{-1}} = (\overline{A})^{-1}$.

Hint. Consider the image of the graph of A under the map

$$\begin{aligned} \chi\colon X \times Y &\to Y \times X \\ (x, y) &\mapsto (y, x). \end{aligned}$$