8.1. A result by Lions-Stampacchia

Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $f\colon H \to \mathbb{R}$ be a continuous linear functional and let $a\colon H \times H \to \mathbb{R}$ be a bilinear map satisfying

(i) $\forall x, y \in H: \quad a(x,y) = a(y,x)$

(ii) $\exists \Lambda > 0 \quad \forall x, y \in H: \quad |a(x,y)| \leq \Lambda \|x\|_H \|y\|_H$

(iii) $\exists \lambda > 0 \quad \forall x \in H: \quad a(x,x) \geq \lambda \|x\|_H^2.$

Consider the functional $J\colon H \to \mathbb{R}$ given by $J(x) = a(x,x) - 2f(x)$ and prove that there exists a unique $y_0 \in K$ such that the two following inequalities both hold.

$$\begin{aligned} \forall y \in K: &\quad J(y_0) \leq J(y), \\ \forall y \in K: &\quad a(y_0, y - y_0) \geq f(y - y_0). \end{aligned}$$

8.2. Duality of sequence spaces

Consider the spaces

$$c_0 := \Big\{ (x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0 \Big\}, \qquad c := \Big\{ (x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists} \Big\}.$$

(a) Quick warm-up: Is $(c_0, \|\cdot\|_{\ell^\infty})$ a Banach space? Is $(c, \|\cdot\|_{\ell^\infty})$ a Banach space?

(b) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(c) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

8.3. Projection to convex sets

Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $P\colon H \to K$ be the operator which maps $x \in H$ to the unique point $Px \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ which was constructed in problem 7.7 (b).

(a) For every $x_1, x_2 \in H$ prove the inequality

$$\|Px_1 - Px_2\|_H \leq \|x_1 - x_2\|_H.$$

(b) Prove that

$$K = \bigcap_{x\in H} \{ y \in H \mid (x - Px, Px - y)_H \geq 0 \}.$$

8.4. Strict convexity

Definition. A normed space $(X, \|\cdot\|_X)$ is called *strictly convex* if $\|\lambda x + (1-\lambda)y\|_X < 1$ holds for all $0 < \lambda < 1$ and all $x, y \in X$ with $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$.

Let $(X, \|\cdot\|_X)$ be a normed space. The "abundance"-Lemma (Satz 4.2.1) states that

$$\forall x \in X \quad \exists x^* \in X^* : \quad \|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2.$$

(a) Prove that if X^* (but not necessarily X) is strictly convex, then for all $x \in X$ there exists a *unique* $x^* \in X^*$ with $\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2$.

(b) Find a counterexample for uniqueness of such x^*, if X^* is not strictly convex.

8.5. Functional on the span of a sequence

Let $(X, \|\cdot\|_X)$ be a normed space, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X and $(\alpha_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Prove that the following statements are equivalent.

(i) There exists $\ell \in X^*$ satisfying $\ell(x_k) = \alpha_k$ for every $k \in \mathbb{N}$.

(ii) There exists $\gamma > 0$ such that for every sequence $(\beta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ and all $n \in \mathbb{N}$

$$\left|\sum_{k=1}^{n} \beta_k \alpha_k\right| \leq \gamma \left\|\sum_{k=1}^{n} \beta_k x_k\right\|_X.$$