10.1. Project: The weak topology is not metrizable

Definition. Let (X,τ) be a topological space. Denoting the set of all neighbourhoods of a point $x\in X$ by

$$\mathcal{U}_x=\{U\subset X\mid \exists\, O\in\tau:\ x\in O\subset U\},$$

we call $\mathcal{B}_x\subset\mathcal{U}_x$ a *neighbourhood basis* of x in (X,τ), if $\forall\, U\in\mathcal{U}_x\ \ \exists V\in\mathcal{B}_x:\ V\subset U$.

Definition. A topological space (X,τ) is called *metrizable* if there exists a metric (namely a distance function) $d\colon X\times X\to\mathbb{R}$ on X (as defined in problem 1.1) such that, denoting $B_\varepsilon(a)=\{x\in X\mid d(x,a)<\varepsilon\}$, there holds

$$\tau=\{O\subset X\mid \forall a\in O\ \exists\,\varepsilon>0:\ B_\varepsilon(a)\subset O)\}.$$

(a) Show that any metrizable topology τ satisfies the *first axiom of countability* which means that each point has a *countable* neighbourhood basis.

In the following, $(X,\|\cdot\|_X)$ is a normed space and τ_{w} denotes the weak topology on X.

(b) Prove that

$$\mathcal{B}:=\left\{\bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon,\varepsilon)\Big)\ \middle|\ n\in\mathbb{N},\ f_1,\dots,f_n\in X^*,\ \varepsilon>0\right\}$$

is a neighbourhood basis of $0\in X$ in (X,τ_{w}).

(c) Prove the following lemma: Let $f_1,\dots,f_n\in X^*$ and $f\in X^*$ be given. Let

$$N:=\{x\in X\mid f_1(x)=\ldots=f_n(x)=0\}.$$

Then $f(x)=0$ for every $x\in N$ if and only if $f=\lambda_1 f_1+\ldots+\lambda_n f_n$ for some $\lambda_1,\dots,\lambda_n\in\mathbb{R}$.

(d) Using (b) and (c), show that if (X,τ_{w}) were first countable, then $(X^*,\|\cdot\|_{X^*})$ would admit a countable algebraic basis.

(e) Assume that the normed space $(X,\|\cdot\|_X)$ is infinite-dimensional and conclude from (a), (d) and problem 4.1 (a) that the topological space (X,τ_{w}) is not metrizable.

10.2. Sequential closure

Let X be a set and τ a topology on X. Given a subset $\Omega \subset X$, we use the notation

$$\overline{\Omega}_\tau := \bigcap_{\substack{A \supset \Omega, \\ X \setminus A \in \tau}} A$$

for the closure of Ω in the topology τ and

$$\overline{\Omega}_{\tau\text{-seq}} := \{x \in X \mid \exists (x_n)_{n\in\mathbb{N}} \text{ in } \Omega : \ x_n \xrightarrow{\tau} x \text{ as } n \to \infty\}$$

for the sequential closure of Ω induced by the topology τ, which is based on the notion of convergence in topological spaces:

$$(x_n \xrightarrow{\tau} x) \quad \Leftrightarrow \quad (\forall U \in \tau, \ x \in U \quad \exists N \in \mathbb{N} \quad \forall n \geq N : \quad x_n \in U).$$

(a) Prove that if $A \subset X$ is closed, then A is sequentially closed. Prove the inclusion $\overline{\Omega}_{\tau\text{-seq}} \subset \overline{\Omega}_\tau$ for any subset $\Omega \subset X$.

(b) Let $(X, \tau) = (\ell^2, \tau_{\mathrm{w}})$, where τ_{w} denotes the weak topology on ℓ^2. Find a set $\Omega \subset \ell^2$ for which the inclusion $\overline{\Omega}_{\text{w-seq}} \subset \overline{\Omega}_{\mathrm{w}}$ proven in (a) is strict.

10.3. Convex hull

Definition. Let $(X, \|\cdot\|_X)$ be a normed space. The *convex hull* of $A \subset X$ is defined as

$$\operatorname{conv}(A) := \bigcap_{\substack{B \supset A, \\ B \text{ convex}}} B$$

(a) Prove the following representation theorem for convex hulls

$$\operatorname{conv}(A) = \Big\{ \sum_{k=1}^n \lambda_k x_k \ \Big| \ n \in \mathbb{N}, \ x_1, \ldots, x_n \in A, \ \lambda_1, \ldots, \lambda_n \geq 0, \ \sum_{k=1}^n \lambda_k = 1 \Big\}.$$

(b) Using part (a), prove Mazur's Lemma: If $(x_k)_{k\in\mathbb{N}}$ is a sequence in X satisfying $x_k \xrightarrow{\mathrm{w}} x$ as $k \to \infty$, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of convex linear combinations

$$y_n = \sum_{k=1}^n a_{kn} x_k, \quad n \in \mathbb{N}, \quad a_{1n}, \ldots, a_{kn} \geq 0, \quad \sum_{k=1}^n a_{kn} = 1,$$

such that $\|y_n - x\|_X \to 0$ as $n \to \infty$.

(c) Let $(X, \|\cdot\|_X)$ be a normed space and let $A, B \subset X$ be compact, convex subsets. Using part (a), prove that $\operatorname{conv}(A \cup B)$ is compact.

10.4. Non-compactness

In each of the Banach spaces below, find a sequence which is bounded but does not have a convergent subsequence.

(a) $\left(L^p([0,1]), \|\cdot\|_{L^p([0,1])}\right)$ for $1 \leq p \leq \infty$,

(b) $(c_0, \|\cdot\|_{\ell^\infty})$ where $c_0 \subset \ell^\infty$ is the space of sequences converging to zero.

10.5. Separability

Let $(X, \|\cdot\|_X)$ be a normed space. Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.