11.1. Dual Operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Recall that if $T \in L(X,Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\forall x \in X \quad \forall y^* \in Y^*: \quad \langle T^* y^*, x\rangle_{X^* \times X} = \langle y^*, Tx\rangle_{Y^* \times Y}.$$

Prove the following facts about dual operators.

(a) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$

(b) If $T \in L(X,Y)$ and $S \in L(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(c) If $T \in L(X,Y)$ is bijective with inverse $T^{-1} \in L(Y,X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(d) Let $\mathcal{I}_X \colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y \colon Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X,Y): \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

11.2. Isomorphisms and Isometries

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X,Y)$. Prove the following.

(a) If T is an isomorphism with $T^{-1} \in L(Y,X)$, then T^* is an isomorphism.

(b) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

(c) If X and Y are both reflexive, then the reverse implications of (a) and (b) hold.

(d) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

11.3. Minimal Energy

Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded subset. For $g \in L^2(\mathbb{R}^m)$, we define the map

$$\begin{aligned} V\colon L^2(\Omega) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y)\, dy\, dx \end{aligned}$$

and given $h \in L^2(\Omega)$, we define the map

$$\begin{aligned} E\colon L^2(\Omega) &\to \mathbb{R} \\ f &\mapsto \|f-h\|^2_{L^2(\Omega)} + V(f). \end{aligned}$$

(a) Prove that V is weakly sequentially continuous.

(b) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega) := \{ f \in L^2(\Omega) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega\}$$

attains a global minimum.

11.4. Compact Operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact}\}$$

the set of *compact operators* between X and Y. Prove the following statements.

(a) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

(b) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(c) Let $T \in L(X,Y)$. If its range $T(X) \subset Y$ is finite-dimensional, then $T \in K(X,Y)$.

(d) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is a compact operator, then $S \circ T$ is a compact operator.

(e) If X is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to norm-convergent sequences is a compact operator.