1.1. Equivalent Norms

(a) Let $n = \dim X$ and let $\{e_1, \dots, e_n\}$ be a basis for X. Then every $x \in X$ is of the form $x = \sum_{k=1}^n x_k e_k$ with uniquely determined components $x_1, \dots, x_n \in \mathbb{R}$. Recall that

$$\|x\|_\infty := \max_{k\in\{1,\dots,n\}} |x_k|$$

defines a norm on X. We show that any given norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$ and therefore any two norms are equivalent to each other. We have

$$\begin{aligned}\|x\| = \Big\|\sum_{k=1}^n x_k e_k\Big\| &\leq \sum_{k=1}^n \|x_k e_k\| = \sum_{k=1}^n |x_k| \|e_k\| \\ &\leq n\Big(\max_{k\in\{1,\dots,n\}} |x_k|\Big)\Big(\max_{k\in\{1,\dots,n\}} \|e_k\|\Big) = nM\|x\|_\infty \qquad (*)\end{aligned}$$

where

$$M := \Big(\max_{k\in\{1,\dots,n\}} \|e_k\|\Big)$$

is a finite constant. The triangle inequality implies $\big|\|x\| - \|y\|\big| \leq \|x - y\|$. Combined with $(*)$ we have

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \leq nM\|x - y\|_\infty$$

for every $x, y \in X$. This implies that $\|\cdot\| \colon (X, \|\cdot\|_\infty) \to \mathbb{R}$ is a continuous map. We restrict this map to $K := \{x \in X \mid \|x\|_\infty = 1\}$ which is a closed and bounded subset in a finite-dimensional space, hence compact. Therefore, the function $\|\cdot\|$ attains its extrema on K. Let $x_1, x_2 \in X$ such that

$$m_1 := \min_{x\in K} \|x\| = \|x_1\|, \qquad m_2 := \max_{x\in K} \|x\| = \|x_2\|$$

Since $\|x_1\|_\infty = 1$ we have $x_1 \neq 0$ and $m_1 > 0$. For arbitrary $x \in X \setminus \{0\}$ we have

$$\Big(\frac{1}{\|x\|_\infty} x\Big) \in K \quad \Rightarrow \quad 0 < m_1 \leq \Big\|\frac{1}{\|x\|_\infty} x\Big\| \leq m_2 < \infty.$$

Multiplication with $\|x\|_\infty$ implies

$$0 < m_1\|x\|_\infty \leq \|x\| \leq m_2\|x\|_\infty < \infty.$$

Any other given norm $\|\cdot\|'$ satisfies analogously

$$0 < m_1'\|x\|_\infty \leq \|x\|' \leq m_2'\|x\|_\infty < \infty.$$

The combination of the two last inequalities proves

$$\exists C > 0 \ \forall x \in X : \quad C^{-1}\|x\|' \leq \|x\| \leq C\|x\|'.$$

(b) Let d be the metric on $\mathbb{R}^2$ induced by the Euclidean norm. We define d' on $\mathbb{R}^2$ by

$$d'(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

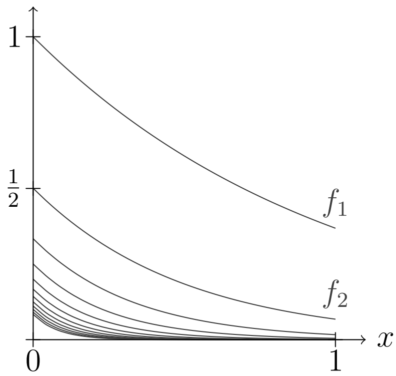
Let z be a point on the Euclidean unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ let $z_n = \frac{1}{n}z$. Then, $d(0,z_n) = \frac{1}{n}$ and $d'(0,z_n) = 1$. Since an inequality of the form $1 \leq C\frac{1}{n}$ can not hold for every $n \in \mathbb{N}$ if C is finite, d and d' are not equivalent.

(c) Let $X = C^1([0,1])$. Let $\|\cdot\|$ and $\|\cdot\|'$ be the two norms on X given by

$$\|u\| := \|u\|_{C^0} = \sup_{x\in[0,1]} |u(x)|, \qquad \|u\|' := \max\Big\{\sup_{x\in[0,1]} |u(x)|, \sup_{x\in[0,1]} |u'(x)|\Big\}$$

For $n \in \mathbb{N}$ we consider

$$f_n \colon [0,1] \to \mathbb{R}$$
$$x \mapsto \frac{e^{-nx}}{n}.$$



Then, $f_n \in C^1([0,1])$ for every $n \in \mathbb{N}$. Moreover, $\|f_n\| = \frac{1}{n}$ and $\|f_n\|' = 1$. Since an inequality of the form $1 \leq C\frac{1}{n}$ can not hold for every $n \in \mathbb{N}$ if C is finite, $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent.

1.2. Intrinsic Characterisations

(a) If the norm $\|\cdot\|$ is induced by the scalar product $\langle\cdot,\cdot\rangle$, then the parallelogram identity holds:

$$\begin{aligned}
&\|x+y\|^2 + \|x-y\|^2 \\
&= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

Conversely, we assume that $\|\cdot\|$ satisfies the parallelogram identity and claim that

$$\langle x,y\rangle := \tfrac{1}{4}\|x+y\|^2 - \tfrac{1}{4}\|x-y\|^2$$

defines an scalar product which induces $\|\cdot\|$.

• *Symmetry.* Since $\|x-y\| = \|(-1)(y-x)\| = \|y-x\|$ and since $x+y = y+x$, we have $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$.

• *Linearity.* Let $x, y, z \in V$. We use the parallelogram identity in the following way.

$$\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2.$$

We rewrite the equation above to obtain

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 =: A$$

and switch the roles of x and y to get

$$\|x+y+z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2 =: B.$$

Therefore,

$$\begin{aligned}\|x+y+z\|^2 &= \frac{A}{2} + \frac{B}{2} \\ &= \|x+z\|^2 + \|y\|^2 + \|y+z\|^2 + \|x\|^2 - \frac{\|x-y+z\|^2 + \|y-x+z\|^2}{2}.\end{aligned} \tag{1}$$

Analogously,

$$\begin{aligned}&\|x+y-z\|^2 \\ &= \|x-z\|^2 + \|y\|^2 + \|y-z\|^2 + \|x\|^2 - \frac{\|x-y-z\|^2 + \|y-x-z\|^2}{2}.\end{aligned} \tag{2}$$

Note that the last term of (1) agrees with the last term of (2). Hence, we have

$$\begin{aligned}\langle x+y, z\rangle &= \tfrac{1}{4}\|x+y+z\|^2 - \tfrac{1}{4}\|x+y-z\|^2 \\ &= \tfrac{1}{4}\Big(\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 - \|y-z\|^2\Big) = \langle x, z\rangle + \langle y, z\rangle.\end{aligned}$$

Let $n \in \mathbb{N}$. By induction on the number of summands in the first slot, we have

$$\langle nx, z\rangle = \Big\langle \sum_{k=1}^{n} x, z\Big\rangle = \sum_{k=1}^{n}\langle x, z\rangle = n\langle x, z\rangle$$

Moreover, since $\langle 0, y\rangle = \frac{1}{4}\Big(\|y\|^2 - \|y\|^2\Big) = 0$,

$$0 = \langle 0, y\rangle = \langle x-x, y\rangle = \langle x, y\rangle + \langle -x, y\rangle \qquad \Rightarrow \ \langle -x, y\rangle = -\langle x, y\rangle.$$

Consequently, $\langle mx, z\rangle = m\langle x, z\rangle$ for every $m \in \mathbb{Z}$. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,

$$\Big\langle \frac{m}{n}x, z\Big\rangle = \frac{n}{n}m\Big\langle \frac{1}{n}x, z\Big\rangle = \frac{m}{n}\Big\langle \frac{n}{n}x, z\Big\rangle = \frac{m}{n}\langle x, z\rangle,$$

which implies $\langle qx, z\rangle = q\langle x, z\rangle$ for every $q \in \mathbb{Q}$.

Let $\lambda \in \mathbb{R}$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to λ for $n \to \infty$. Since the triangle inequality $|\|x\| - \|y\|| \leq \|x - y\|$ implies that the norm is a continuous map, we have

$$\begin{aligned}\langle \lambda x, z\rangle &= \tfrac{1}{4}\|\lambda x + z\|^2 - \tfrac{1}{4}\|\lambda x - z\|^2 = \lim_{n\to\infty}\Big(\tfrac{1}{4}\|q_n x + z\|^2 - \tfrac{1}{4}\|q_n x - z\|^2\Big)\\ &= \lim_{n\to\infty}\langle q_n x, z\rangle = \lim_{n\to\infty} q_n\langle x, z\rangle = \lambda\langle x, z\rangle.\end{aligned}$$

Linearity in the second argument follows by symmetry.

- *Positive-definiteness.* For all $x \in V$, we have

$$\langle x, x\rangle = \tfrac{1}{4}\|x + x\|^2 - \tfrac{1}{4}\|x - x\|^2 = \tfrac{1}{4}\|2x\|^2 = \|x\|^2 \geq 0.$$

This also shows that $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$. Moreover, $\langle x, x\rangle = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$.

(b) If the metric d is induced by the norm $\|\cdot\|$, then

$$\begin{aligned}d(x+v, y+v) &= \|(x+v) - (y+v)\| = \|x - y\| = d(x,y),\\ d(\lambda x, \lambda y) &= \|\lambda x - \lambda y\| = \|\lambda(x-y)\| = |\lambda|\|x - y\|.\end{aligned}$$

Conversely, we assume that the metric d is translation invariant and homogeneous and claim that

$$\|x\| := d(x, 0)$$

defines a norm which induces d. The function $\|\cdot\|$ is indeed a norm, because for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}\|x\| = 0 \;&\Leftrightarrow\; d(x,0) = 0 \;\Leftrightarrow\; x = 0,\\ \|\lambda x\| &= d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x, 0) = |\lambda|\|x\|,\\ \|x + y\| &= d(x+y, 0) \leq d(x+y, y) + d(y, 0) = d(x, 0) + d(y, 0) = \|x\| + \|y\|.\end{aligned}$$

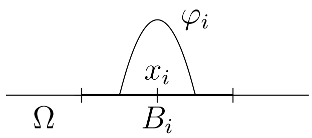
Moreover, $\|\cdot\|$ induces the metric d since for all $x, y \in V$

$$\|x - y\| = d(x - y, 0) = d(x, y).$$

1.3. Infinite-dimensional vector spaces and separability

(a) Suppose by contradiction, $L^p(\Omega)$ has finite dimension $d \in \mathbb{N}$. Since $\emptyset \neq \Omega \subset \mathbb{R}^n$ is open there exist $d+1$ disjoint balls $B_i := B_{r_i}(x_i) \subset \Omega$ for $i = 1, \ldots, d+1$. Let

$$\varphi_i(x) = \max\Big\{0, 1 - \frac{4|x - x_i|^2}{r_i^2}\Big\}.$$



Then, $\varphi_1, \ldots, \varphi_{d+1} \in C_c(\Omega) \subset L^p(\Omega)$ with disjoint supports. Moreover, since the subset $\{\varphi_1, \ldots, \varphi_{d+1}\}$ contains more than d Elements, it must be linearly dependent. Let $\lambda_1, \ldots, \lambda_{d+1} \in \mathbb{R}$ be not all equal to 0 such that

$$\sum_{i=1}^{d+1} \lambda_i \varphi_i = 0.$$

However, if we multiply by φ_j for any $j \in \{1, \ldots, d+1\}$ and integrate over Ω,

$$0 = \int_\Omega \sum_{i=1}^{d+1} \lambda_i \varphi_i \varphi_j \,\mathrm{d}\mu = \int_\Omega \lambda_i \varphi_j^2 \,\mathrm{d}\mu = \lambda_j \int_\Omega \varphi_j^2 \,\mathrm{d}\mu \qquad \Rightarrow \lambda_j = 0.$$

(b) We define $I_n := (\frac{1}{n+1}, \frac{1}{n}) \subset (0,1)$ for $n \in \mathbb{N}$ and consider the characteristic function χ_{I_n} of I_n, i. e.

$$\chi_{I_n}(x) := \begin{cases} 1, & \text{if } x \in I_n, \\ 0, & \text{if } x \in (0,1) \setminus I_n. \end{cases}$$

Given any subset $\emptyset \neq M \subset \mathbb{N}$ we define the function $f_M \in L^\infty((0,1))$ by

$$f_M(x) := \sum_{n \in M} \chi_{I_n}(x)$$

Since the intervals I_n are pairwise disjoint, open and non-empty, we have $\|f_M\|_{L^\infty} = 1$ for every $\emptyset \neq M \subset \mathbb{N}$. For the same reason,

$$\|f_M - f_{M'}\|_{L^\infty} = 1.$$

if $M \neq M'$. Therefore, the balls $B_M = \{g \in L^\infty((0,1)) \mid \|g - f_M\|_{L^\infty} < \frac{1}{3}\}$ are pairwise disjoint. If $S \subset L^\infty((0,1))$ is any dense subset, then $S \cap B_M \neq \emptyset$ for every $\emptyset \neq M \subset \mathbb{N}$. Thus, there is a surjective map $S \to \{B_M \mid \emptyset \neq M \subset \mathbb{N}\}$. Since there are uncountably many different subsets of $\mathbb{N}$, the set S must be uncountable as well. Therefore, $L^\infty((0,1))$ does not admit a countable dense subset.