2.1. A metric on sequences

(a) The function $d\colon S \times S \to \mathbb{R}$ is well-defined because for any $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in S$,

$$0 \leq d\Big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\Big) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \leq \sum_{n\in\mathbb{N}} 2^{-n} < \infty.$$

Symmetry and the requirement $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0 \Leftrightarrow \forall n \in \mathbb{N} : x_n = y_n$ both follow from the respective property of $|x_n - y_n|$. It remains to prove the triangle inequality. For every $x, y, z \in \mathbb{R}$ we observe

$$\begin{aligned}
\frac{|x-z|}{1+|x-z|} &= 1 - \frac{1}{1+|x-z|} \\
&\leq 1 - \frac{1}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\
&= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\
&\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}.
\end{aligned}$$

Replacing x, y, z by x_n, y_n, z_n in the definition of d proves the triangle inequality.

(b) Completeness of (S, d) will follow from the following two claims.

Claim 1. If $(X^{(k)})_{k\in\mathbb{N}} = ((x_n^{(k)})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ is a Cauchy-sequence in (S, d), then $(x_n^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ for every $n \in \mathbb{N}$.

Proof. Let $n_0 \in \mathbb{N}$ be arbitrary but fixed. Let $0 < \varepsilon < 1$. By assumption there exists $N \in \mathbb{N}$ such that for every $k, \ell > N$

$$d(X^{(k)}, X^{(\ell)}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n^{(k)} - x_n^{(\ell)}|}{1 + |x_n^{(k)} - x_n^{(\ell)}|} < \frac{\varepsilon}{2^{n_0}(1+\varepsilon)}.$$

In particular, since every summand is non-negative,

$$2^{-n_0} \frac{|x_{n_0}^{(k)} - x_{n_0}^{(\ell)}|}{1 + |x_{n_0}^{(k)} - x_{n_0}^{(\ell)}|} < \frac{\varepsilon}{2^{n_0}(1+\varepsilon)}.$$

This implies $|x_{n_0}^{(k)} - x_{n_0}^{(\ell)}| < \varepsilon$ for every $k, \ell > N$ which means that $(x_{n_0}^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence. □

Claim 2. Let $(X^{(k)})_{k\in\mathbb{N}} = ((x_n^{(k)})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ be a sequence in S. If $x_n^{(k)} \xrightarrow{k\to\infty} x_n$ in $(\mathbb{R},|\cdot|)$ for every $n\in\mathbb{N}$, then $X^{(k)} \xrightarrow{k\to\infty} X = (x_n)_{n\in\mathbb{N}}$ in (S,d).

Proof. Let $\varepsilon > 0$ and let $N_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=N_\varepsilon+1}^{\infty} 2^{-n} = 2^{-N_\varepsilon} \leq \frac{\varepsilon}{2}.$$

By assumption, there exists $N\in\mathbb{N}$ such that $|x_n^{(k)} - x_n| < \frac{\varepsilon}{2}$ for every $k \geq N$ and all the finitely many $n \in \{1,\dots,N_\varepsilon\}$. Hence

$$\begin{aligned} d(X^{(k)},X) &= \sum_{n=1}^{N_\varepsilon} 2^{-n}\frac{|x_n^{(k)}-x_n|}{1+|x_n^{(k)}-x_n|} + \sum_{n=N_\varepsilon+1}^{\infty} 2^{-n}\frac{|x_n^{(k)}-x_n|}{1+|x_n^{(k)}-x_n|} \\ &\leq \frac{\varepsilon}{2}\sum_{n=1}^{N_\varepsilon} 2^{-n} + \sum_{n=N_\varepsilon+1}^{\infty} 2^{-n} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ □

Let $(X^{(k)})_{k\in\mathbb{N}} = ((x_n^{(k)})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ be a Cauchy-sequence in (S,d). Since $(\mathbb{R},|\cdot|)$ is complete, $(x_n^{(k)})_{k\in\mathbb{N}}$ has a limit $x_n\in\mathbb{R}$ for every $n\in\mathbb{N}$ by Claim 1. Then, by Claim 2, $(X^{(k)})_{k\in\mathbb{N}}$ converges to $(x_n)_{n\in\mathbb{N}}$ in (S,d). Therefore, (S,d) is complete.

(c) Given arbitrary $s = (s_n)_{n\in\mathbb{N}} \in S$ and $r > 0$, we consider the ball

$$B_r(s) := \{x \in S \mid d(x,s) < r\}.$$

The claim is $S_c \cap B_r(s) \neq \emptyset$. Let $N_r \in \mathbb{N}$ such that

$$\sum_{n=N_r+1}^{\infty} 2^{-n} = 2^{-N_r} < r$$

and let $x = (x_n)_{n\in\mathbb{N}} \in S_c$ be given by

$$x_n = \begin{cases} s_n, & \text{for } n \in \{1,\dots,N_r\}, \\ 0, & \text{for } n > N_r. \end{cases}$$

Then

$$d(x,s) = \sum_{N_r+1}^{\infty} 2^{-n}\frac{|s_n - 0|}{1+|s_n-0|} \leq \sum_{n=N_r+1}^{\infty} 2^{-n} < r.$$

Hence $x \in S_c \cap B_r(s)$. Since $s\in S$ and $r>0$ were arbitrary, it follows that S_c is dense in (S,d).

2.2. A metric on $C^0(\mathbb{R}^m)$

Recall that $K_1 \subset K_2 \subset \ldots \subset \mathbb{R}^m$ is a given family of compact sets such that $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^m$.

(a) The solution is identical to the solution of Problem 2.1 (a) after replacing $|\cdot|$ by $\|\cdot\|_{C^0(K_n)}$ in each summand.

(b) In the following, the restriction of $f \in C^0(\mathbb{R}^m)$ to $K \subset \mathbb{R}^m$ is denoted by $f|_K$.

Claim 1. If $(f_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $(C^0(\mathbb{R}^m), d)$, then $(f_k|_{K_n})_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\big(C^0(K_n), \|\cdot\|_{C^0(K_n)}\big)$ for each of the compact sets $K_n \subset \mathbb{R}^m$.

Claim 2. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C^0(\mathbb{R}^m)$ and let $f \in C^0(\mathbb{R}^m)$. If $f_k|_{K_n} \xrightarrow{k\to\infty} f|_{K_n}$ in $C^0(K_n)$ for every $n \in \mathbb{N}$, then $f_k \xrightarrow{k\to\infty} f$ in $(C^0(\mathbb{R}^m), d)$.

Proof. The proofs are identical to the proofs of Claim 1 and 2 in 2.1 (b) after replacing x_n^k by $f_k|_{K_n}$ and $|\cdot|$ by $\|\cdot\|_{C^0(K_n)}$ in each summand. □

Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy-sequence in $(C^0(\mathbb{R}^m), d)$. Since $\big(C^0(K_n), \|\cdot\|_{C^0(K_n)}\big)$ is complete, $(f_k|_{K_n})_{k\in\mathbb{N}}$ has a limit $g_n \in C^0(K_n)$ for every $n \in \mathbb{N}$ by Claim 1. In particular, given any $n \in \mathbb{N}$ pointwise convergence $f_k(x) \to g_n(x)$ holds for every $x \in K_n$. Since the pointwise limit is unique, $g_{n+1}|_{K_n} = g_n$ for every $n \in \mathbb{N}$. Therefore, there exists a well-defined function $f\colon \mathbb{R}^m \to \mathbb{R}$ such that $g_n = f|_{K_n}$.

Because $K_n \subset K_{n+1}^\circ$ and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^m$, every point $x \in \mathbb{R}^m$ has a neighbourhood on which f inherits the continuity of g_n for some $n \in \mathbb{N}$. Thus, $f \in C^0(\mathbb{R}^m)$. Then, by Claim 2, $(f_k)_{k\in\mathbb{N}}$ converges to f in $(C^0(\mathbb{R}^m), d)$. Therefore, $(C^0(\mathbb{R}^m), d)$ is complete.

(c) Let $f \in C^0(\mathbb{R}^m)$ be arbitrary. Since K_n is compact with $K_n \subset K_{n+1}^\circ$, we have

$$\varepsilon_n := \operatorname{dist}\Big(K_n, (K_{n+1}^\circ)^\complement\Big) > 0.$$

For every $n \in \mathbb{N}$ we define the function $\varphi \in C_c^0(\mathbb{R}^m)$ by

$$\varphi_n(x) = \begin{cases} 1 - \frac{1}{\varepsilon_n}\operatorname{dist}(x, K_n), & \text{if } \operatorname{dist}(x, K_n) \leq \varepsilon_n, \\ 0, & \text{else} \end{cases}$$

and consider $f_k := \varphi_k f$. By construction, $f_k|_{K_n} \xrightarrow{k\to\infty} f|_{K_n}$ for every $n \in \mathbb{N}$. Therefore, $f_k \xrightarrow{k\to\infty} f$ in $(C^0(\mathbb{R}^m), d)$ by Claim 2 of (b). Since $f_k \in C_c^0(\mathbb{R}^m)$ for every $k \in \mathbb{N}$ and since $f \in C^0(\mathbb{R}^m)$ was arbitrary, we have shown that $C_c^0(\mathbb{R}^m)$ is dense in $(C^0(\mathbb{R}^m), d)$.

2.3. Statements of Baire

For a metric space (M, d) we shall prove equivalence of

(i) Every residual set $\Omega \subset M$ is dense in M.
(ii) The interior of every meagre set $A \subset M$ is empty.
(iii) The empty set is the only subset of M which is open and meagre.
(iv) Countable intersections of dense open sets are dense.

"(i) $\Rightarrow$ (ii)" Let $A \subset M$ be a meagre set. Then, $A^{\complement}$ is residual and dense in M by (i). Hence, $\emptyset = (M \setminus A^{\complement})^{\circ} = A^{\circ}$.

"(ii) $\Rightarrow$ (iii)" Let $A \subset M$ be open and meagre. Then $A = A^{\circ}$ and $A^{\circ} = \emptyset$ by (ii).

"(iii) $\Rightarrow$ (iv)" Let $A = \bigcap_{n\in\mathbb{N}} A_n$ be a countable intersection of dense open sets $A_n \subset M$. Since A_n is dense, $(A_n^{\complement})^{\circ} = \emptyset$. Since A_n is open, $A_n^{\complement}$ is closed. Thus, $(\overline{A_n^{\complement}})^{\circ} = (A_n^{\complement})^{\circ} = \emptyset$, which means that $A_n^{\complement}$ is nowhere dense. Thus, $A^{\complement} = \bigcup_{n\in\mathbb{N}} A_n^{\complement}$ is meagre. $(A^{\complement})^{\circ}$ is open and meagre, hence empty by (iii). This implies that A is dense in M.

"(iv) $\Rightarrow$ (i)" Let $\Omega \subset M$ be a residual set. Since $A = \Omega^{\complement}$ is meagre, $A = \bigcup_{n\in\mathbb{N}} A_n$ for nowhere dense sets A_n. Then $\emptyset = (\overline{A_n})^{\circ} = (M \setminus (\overline{A_n})^{\complement})^{\circ}$ which implies that $(\overline{A_n})^{\complement}$ is dense in M. Moreover, $(\overline{A_n})^{\complement}$ is open since $\overline{A_n}$ is closed. Then, (iv) implies density of

$$\Omega = A^{\complement} = \bigcap_{n\in\mathbb{N}} A_n^{\complement} \supseteq \bigcap_{n\in\mathbb{N}} (\overline{A_n})^{\complement}.$$

2.4. Discrete L^p-spaces and inclusions

(a) Let $1 \leq p < q \leq \infty$. It suffices to prove the inequality $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ for all $x \in \ell^p$ which implies the inclusion $\ell^p \subset \ell^q$ by definition of the spaces. Since $(n^{-\frac{1}{p}})_{n\in\mathbb{N}} \in \ell^q \setminus \ell^p$, the inclusion is strict.

Scaling. Since $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ if and only if $\|\lambda x\|_{\ell^q} \leq \|\lambda x\|_{\ell^p}$ for some $\lambda > 0$, it suffices to prove $\|x\|_{\ell^q} \leq 1$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ with $\|x\|_{\ell^p} = 1$.

Case $q = \infty$. For all $n \in \mathbb{N}$ we have

$$|x_n| = \Bigl(|x_n|^p\Bigr)^{\frac{1}{p}} \leq \Bigl(\sum_{k=1}^{\infty} |x_k|^p\Bigr)^{\frac{1}{p}} = \|x\|_{\ell^p} = 1.$$

Therefore, $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n| \leq 1$.

Case $q < \infty$. The assumption $\|x\|_{\ell^p} = 1$ implies $|x_n| \leq 1$ for all $n \in \mathbb{N}$. Since $1 \leq p < q$, we have $|x_n|^q \leq |x_n|^p$ for all $n \in \mathbb{N}$. This implies the inequality

$$\|x\|_{\ell^q} = \Bigl(\sum_{n\in\mathbb{N}} |x_n|^q\Bigr)^{\frac{1}{q}} \leq \Bigl(\sum_{n\in\mathbb{N}} |x_n|^p\Bigr)^{\frac{1}{q}} = \Bigl(\|x\|_{\ell^p}^p\Bigr)^{\frac{1}{q}} = 1^{\frac{p}{q}} = 1.$$

(b) In order to show that $A_n = \{x \in \ell^q \mid \|x\|_{\ell^p} \leq n\}$ is closed in $(\ell^q, \|\cdot\|_{\ell^q})$, we will prove that the limit of every ℓ^q-convergent sequence with elements in A_n is also in A_n.

Let $(a^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements $a^{(k)} = (a_j^{(k)})_{j\in\mathbb{N}} \in A_n$. Suppose $b = (b_j)_{j\in\mathbb{N}} \in \ell^q$ satisfies $\lim\limits_{k\to\infty} \|a^{(k)} - b\|_{\ell^q} = 0$. Then, for every $j \in \mathbb{N}$,

$$|a_j^{(k)} - b_j| \leq \Big(\sum_{i\in\mathbb{N}} |a_i^{(k)} - b_i|^q\Big)^{\frac{1}{q}} = \|a^{(k)} - b\|_{\ell^q} \xrightarrow{k\to\infty} 0.$$

Let $N \in \mathbb{N}$ be arbitrary. By continuity of $|\cdot|^p \colon \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{j=1}^{N} |b_j|^p = \lim_{k\to\infty} \sum_{j=1}^{N} |a_j^{(k)}|^p \leq \limsup_{k\to\infty} \|a^{(k)}\|_{\ell^p}^p \leq n^p$$

since the number of summands is finite. In the limit $N \to \infty$, we see $\|b\|_{\ell^p}^p \leq n^p$, which implies $b \in A_n$. Therefore, A_n is closed in $(\ell^q, \|\cdot\|_{\ell^q})$.

Towards a contradiction, suppose A_n has non-empty interior in the ℓ^q- topology. Then there exist $a = (a_m)_{m\in\mathbb{N}} \in A_n$ and $\varepsilon > 0$ such that

$$B := \{x \in \ell^q \mid \|a - x\|_{\ell^q} < \varepsilon\} \subset A_n.$$

Consider $b = (b_m)_{m\in\mathbb{N}} \in \ell^q$ given by $b_m = m^{-\frac{1}{p}}$. Indeed, $\sum_{m=1}^{\infty} m^{-\frac{q}{p}} < \infty$ since $p < q$. We define $z = (z_m)_{m\in\mathbb{N}}$ by

$$z_m = a_m + \frac{\varepsilon b_m}{2\|b\|_{\ell^q}}.$$

Then $\|a - z\|_{\ell^q} = \frac{\varepsilon}{2}$ and $z \in B$. However, $b \notin \ell^p$ and $a \in \ell^p$ imply $z \notin \ell^p \supset A_n$ which contradicts $B \subset A_n$. Therefore, A_n has empty interior in $(\ell^q, \|\cdot\|_{\ell^q})$.

Being closed with empty interior, A_n is nowhere dense in $(\ell^q, \|\cdot\|_{\ell^q})$. Since $\ell^p = \bigcup_{n\in\mathbb{N}} A_n$ we may conclude that ℓ^p is meagre in ℓ^q.

(c) Since $\ell^{p_1} \subset \ell^{p_2}$ for $p_1 < p_2$ by (a) we have

$$\bigcup_{p\in[1,q[} \ell^p = \bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p.$$

By (b), the right hand side is a countable union of meagre subsets of $(\ell^q, \|\cdot\|_{\ell^q})$ and therefore meagre itself (see lecture notes, Beispiel 1.3.2.iii). Being complete, ℓ^q is not meagre in $(\ell^q, \|\cdot\|_{\ell^q})$. Therefore, we may conclude strict inclusion

$$\bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p \subsetneq \ell^q.$$