4.1. Algebraic basis

(a) Towards a contradiction, we assume that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$. For $n \in \mathbb{N}$ we define the linear subspaces $A_n = \operatorname{span}\{e_1, \ldots, e_n\} \subset X$.

As finite dimensional subspace, A_n is closed. Suppose that A_n has non-empty interior. Then there exist $x \in A_n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_n$. Since A_n is a linear subspace, we may subtract $x \in A_n$ from the elements in $B_\varepsilon(x)$ to obtain $B_\varepsilon(0) \subset A_n$. For the same reason,

$$A_n \supset \{\lambda y \mid \lambda > 0,\ y \in B_\varepsilon(x)\} = X.$$

This however implies $\dim X \leq n$ which contradicts our assumption that the algebraic basis of X is countably infinite. Thus $(\overline{A_n})^\circ = A_n^\circ = \emptyset$ which means that A_n is nowhere dense. By assumption,

$$X = \bigcup_{n \in \mathbb{N}} A_n,$$

which implies that X is meagre – a contradiction to the Baire Lemma stating that X being complete is of second category.

(b) Let X be the space of polynomials $p\colon [0,1] \to \mathbb{R}$ with real coefficients endowed with the norm $\|\cdot\|_{C^0([0,1])}$. Let $f_n\colon [0,1] \to \mathbb{R}$ be given by the monomial $f_n(x) = x^n$. Then, $\{f_n \mid n \in \mathbb{N}\}$ is a countable algebraic basis for X. According to (a), the space $(X, \|\cdot\|_{C^0([0,1])})$ must be incomplete.

4.2. Closed subspaces

Claim 1. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in \ell^1$ and let $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$. Then, the following implication is true.

$$\lim_{k\to\infty} \|x^{(k)} - x\|_{\ell^1} = 0 \quad \Rightarrow \quad \forall n \in \mathbb{N}: \ \lim_{k\to\infty} |x_n^{(k)} - x_n| = 0.$$

Proof. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. By assumption, there exists $K_\varepsilon \in \mathbb{N}$ such that

$$\forall k \geq K_\varepsilon: \quad |x_n^{(k)} - x_n| \leq \sum_{n=1}^{\infty} |x_n^{(k)} - x_n| = \|x^{(k)} - x\|_{\ell^1} < \varepsilon. \qquad \square$$

Claim 2. $U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n} = 0\}$ is closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in U$ converging to $x = (x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{2n}^{(k)} = 0$ for every $n \in \mathbb{N}$. According to Claim 1,

$$x_{2n} = \lim_{k\to\infty} x_{2n}^{(k)} = 0$$

for every $n \in \mathbb{N}$. Thus, $x \in U$. $\square$

Claim 3. $V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n-1} = nx_{2n}\}$ is closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in V$ converging to $x = (x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{2n-1}^{(k)} = nx_{2n}^{(k)}$ for every $n \in \mathbb{N}$. By Claim 1,

$$x_{2n-1} = \lim_{k\to\infty} x_{2n-1}^{(k)} = \lim_{k\to\infty} nx_{2n}^{(k)} = nx_{2n}$$

for every $n \in \mathbb{N}$. Thus, $x \in V$. $\square$

Claim 4. $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\} \subset U \oplus V$.

Proof. Let $x \in c_c$. Then, $x = u + v$ with $u = (u_m)_{m\in\mathbb{N}}$ and $v = (v_m)_{m\in\mathbb{N}}$ given by

$$u_m = \begin{cases} x_m - nx_{m+1}, & \text{if } m = 2n-1, \\ 0, & \text{if } m \text{ is even} \end{cases} \qquad v_m = \begin{cases} nx_{m+1}, & \text{if } m = 2n-1, \\ x_m, & \text{if } m \text{ is even.} \end{cases}$$

The assumption $x \in c_c$ implies $v, u \in c_c \subset \ell^1$. Then, $u \in U$ holds by construction and $v \in V$ follows from $v_{2n-1} = nx_{2n-1+1} = nx_{2n} = nv_{2n}$ for every $n \in \mathbb{N}$. $\square$

Claim 5. The space c_c is dense in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $x \in \ell^1$. Let $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_n^{(k)} = \begin{cases} x_n & \text{for } n < k, \\ 0 & \text{for } n \geq k. \end{cases}$$

Then,

$$\|x^{(k)} - x\|_{\ell^1} = \sum_{n=k}^{\infty} |x_n| \xrightarrow{k\to\infty} 0. \qquad \square$$

Claim 6. The sequence $x = (x_m)_{m\in\mathbb{N}}$ defined as follows is in ℓ^1 but not in $U \oplus V$.

$$x_m = \begin{cases} 0, & \text{if } m \text{ is odd}, \\ \frac{1}{n^2}, & \text{if } m = 2n. \end{cases}$$

Proof. Since $\sum_{n=1}^\infty \frac{1}{n^2} < \infty$ we have $x \in \ell^1$. Suppose $x = u + v$ for $u \in U$ and $v \in V$. Then, $u_{2n} = 0$ implies $v_{2n} = x_{2n} = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. By definition of V, we have $v_{2n-1} = nv_{2n} = \frac{1}{n}$ for every $n \in \mathbb{N}$. However, $\sum_{n=1}^\infty \frac{1}{n} = \infty$ implies $v \notin \ell^1$ which contradicts the definition of V. $\square$

Claims 4, 5 and 6 imply that

$$\overline{U \oplus V} \supset \overline{c_c} = \ell^1 \supsetneq U \oplus V.$$

Therefore, $U \oplus V$ cannot be closed.

4.3. Normal convergence

If $(X, \|\cdot\|)$ is a Banach space, and $(x_k)_{k\in\mathbb{N}}$ any sequence in X with $\sum_{k=1}^{\infty}\|x_k\| < \infty$, then $(s_n)_{n\in\mathbb{N}}$ given by $s_n = \sum_{k=1}^{n} x_k$ is a Cauchy sequence (and hence convergent) since by assumption, every $\varepsilon > 0$ allows $N_\varepsilon \in \mathbb{N}$ such that for every $m \geq n \geq N_\varepsilon$,

$$\|s_m - s_n\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N_\varepsilon+1}^{\infty} \|x_k\| < \varepsilon.$$

Conversely, we assume for every sequence $(x_k)_{k\in\mathbb{N}}$ in X that $\sum_{k=1}^{\infty}\|x_k\| < \infty$ implies convergence of $s_n = \sum_{k=1}^{n} x_k$ in X for $n \to \infty$. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy in X. Then,

$$\forall k \in \mathbb{N} \quad \exists N_k \in \mathbb{N} \quad \forall n, m \geq N_k : \quad \|y_n - y_m\| \leq 2^{-k}.$$

Without loss of generality, we can assume $N_{k+1} > N_k$. Let $x_k := y_{N_{k+1}} - y_{N_k}$. Then,

$$\sum_{k=1}^{\infty}\|x_k\| = \sum_{k=1}^{\infty}\|y_{N_{k+1}} - y_{N_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty,$$

which by assumption implies that

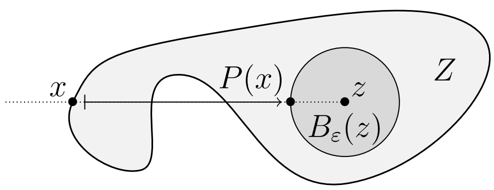
$$s_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n}(y_{N_{k+1}} - y_{N_k}) = y_{N_{n+1}} - y_{N_1}$$

converges in X for $n \to \infty$. Hence, $(y_{N_n})_{n\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to the same limit in X. Thus, X is complete.

4.4. Subsets with compact boundary

If $Z \subset X$ has non-empty interior $Z^\circ \neq \emptyset$, then there exists $z \in Z$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subset Z^\circ$, where $B_\varepsilon(z)$ denotes the ball of radius ε around z in $(X, \|\cdot\|)$ and $\partial B_\varepsilon(z)$ its boundary. We consider the projection

$$\begin{aligned} P\colon Z \setminus \{z\} &\to \partial B_\varepsilon(z) \\ x &\mapsto z + \varepsilon \frac{x - z}{\|x - z\|}. \end{aligned}$$



For every $y \in \partial B_\varepsilon$ the ray $\gamma = \{z + t(y - z) \mid t > 0\}$ must intersect ∂Z since Z is assumed to be bounded. Therefore, $P(\partial Z) = \partial B_\varepsilon(z)$. Being continuous, P maps compact sets onto compact sets. Since ∂Z is assumed to be compact, we have that the sphere $\partial B_\varepsilon(z)$ is compact. This however contradicts the assumption, that the dimension of X is infinite.

4.5. Approaching the sign function

(a) Let $f \in X := C^0([-1,1])$ and $\|\cdot\|_X := \|\cdot\|_{C^0([-1,1])}$. The given map $\varphi\colon X \to \mathbb{R}$ is linear by linearity of the integral. Moreover,

$$|\varphi(f)| \leq \int_0^1 |f(t)| \,\mathrm{d}t + \int_{-1}^0 |f(t)| \,\mathrm{d}t \leq 2\|f\|_{C^0([-1,1])} = 2\|f\|_X$$

implies

$$\|\varphi\|_{L(X,\mathbb{R})} = \sup_{f \in X\setminus\{0\}} \frac{|\varphi(f)|}{\|f\|_X} \leq 2.$$

Since φ is linear, continuity follows from boundedness by Satz 2.2.1.

(b) The sign function $f(x) = \frac{x}{|x|}$ is approximated pointwise by the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in X$ given by

$$f_n(t) = \begin{cases} -1, & \text{for } -1 \leq t < -\frac{1}{n}, \\ nt, & \text{for } -\frac{1}{n} \leq t < \frac{1}{n}, \\ 1, & \text{for } \frac{1}{n} \leq t \leq 1. \end{cases}$$

In particular, $\|f_n\|_X = 1$ for every $n \in \mathbb{N}$. Computing the integrals explicitly, or applying the dominated convergence theorem, we have

$$\lim_{n\to\infty} \varphi(f_n) = 2.$$

(c) Suppose there exists $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$. Since φ is linear, we may assume $\varphi(f) = 2$, otherwise we replace f by $-f$. Then, the estimates

$$\left|\int_0^1 f(t)\,\mathrm{d}t\right| \leq \max_{x\in[-1,1]} |f(x)| = \|f\|_X = 1, \qquad \left|\int_{-1}^0 f(t)\,\mathrm{d}t\right| \leq 1,$$

imply by definition of φ that

$$\int_0^1 f(t)\,\mathrm{d}t = -\int_{-1}^0 f(t)\,\mathrm{d}t = 1. \tag{$*$}$$

Since f is bounded from above by 1 we can conclude from $(*)$ that $f|_{]0,1]} \equiv 1$. In fact, if $f(t^*) < 1$ for some $t^* \in]0,1]$, then $f < 1$ in some neighbourhood of t^* by continuity of f which together with the uniform bound $f \leq 1$ contradicts $(*)$.

Analogously, we conclude $f|_{[-1,0[} \equiv -1$ which (combined with $f|_{]0,1]} \equiv 1$) violates continuity of f at 0.

4.6. Unbounded map and approximations

(a) The operation $T\colon (x_n)_{n\in\mathbb{N}} \mapsto (nx_n)_{n\in\mathbb{N}}$ is linear in each entry and therefore linear as map $T\colon c_c \to c_c$. For every $k\in\mathbb{N}$ we define the sequence $e^{(k)} = (e_n^{(k)})_{n\in\mathbb{N}} \in c_c$ by

$$e_n^{(k)} = \begin{cases} 1, & \text{if } n = k,\\ 0, & \text{otherwise.} \end{cases}$$

Then, $\|e^{(k)}\|_{\ell^\infty} = 1$ for every $k\in\mathbb{N}$ but $\|Te^{(k)}\|_{\ell^\infty} = k$ is unbounded for $k\in\mathbb{N}$. As unbounded linear map, T is not continuous.

(b) For every $m\in\mathbb{N}$ we define

$$\begin{aligned} T_m\colon c_c &\to c_c\\ (x_n)_{n\in\mathbb{N}} &\mapsto (x_1, 2x_2, 3x_3, \ldots, mx_m, 0, 0, \ldots) \end{aligned}$$

Then T_m is linear. $T_m\colon (c_c, \|\cdot\|_\ell^\infty) \to (c_c, \|\cdot\|_\ell^\infty)$ is also bounded for every (fixed) $m\in\mathbb{N}$ since for every $x = (x_n)_{n\in\mathbb{N}} \in c_c$

$$\|T_m x\| = \sup_{n\in\mathbb{N}} |(T_m x)_n| = \max_{n\in\{1,\ldots,m\}} |nx_n| \le m\|x\|_{\ell^\infty}.$$

Hence, T_m is continuous.

Let $x = (x_n)_{n\in\mathbb{N}} \in c_c$ be fixed. Then there exists $N\in\mathbb{N}$ such that $x_n = 0$ for all $n \ge N$ which implies $T_m x = Tx$ for all $m \ge N$. In particular,

$$T_m x \xrightarrow{m\to\infty} Tx.$$