5.1. Operator norm

(a) Let $\langle\cdot,\cdot\rangle$ be the the Euclidean scalar product on $\mathbb{R}^n$ and $|\cdot|$ the Euclidean norm. We choose the standard basis and represent $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ by a matrix which we denote also by A. Let $A^\intercal$ be the transposed matrix. From the definition follows that

$$\begin{aligned}
&\|A\|^2 = \sup\{|Ax|^2 \mid x \in \mathbb{R}^n,\ |x|^2 = 1\},\\
&|Ax|^2 = \langle Ax, Ax\rangle = (Ax)^\intercal(Ax) = x^\intercal A^\intercal Ax = \langle x, A^\intercal Ax\rangle.
\end{aligned}$$

Recall that $A^\intercal A$ is a symmetric matrix and therefore diagonalizable by an orthonormal basis of eigenvectors $e_1,\ldots,e_n$ with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $x \in \mathbb{R}^n$ with $|x|^2 = 1$ be given. Then there exist $x_1,\ldots,x_n \in \mathbb{R}$ such that $x = x_1e_1+\ldots+x_ne_n$ and $x_1^2+\ldots+x_n^2 = 1$. From

$$\langle x, A^\intercal Ax\rangle = \Big\langle x, A^\intercal A\sum_{i=1}^n x_ie_i\Big\rangle = \Big\langle x, \sum_{i=1}^n x_i\lambda_ie_i\Big\rangle = \sum_{i=1}^n \lambda_i x_i^2 \leq \lambda_n\sum_{i=1}^n x_i^2 = \lambda_n$$

we conclude $\|A\|^2 \leq \lambda_n$. Since $\langle e_n, A^\intercal Ae_n\rangle = \langle e_n, \lambda_ne_n\rangle = \lambda_n$, we have $\|A\|^2 = \lambda_n$.

(b) Since A and B are assumed to be symmetric, we have $A^\intercal A = A^2$ and $B^\intercal B = B^2$. In the basis $\mathcal{B}$ respectively $\mathcal{B}'$ we see that $(2017)^2$ is the largest eigenvalue of A^2 respectively B^2. Using (a), we have $\|A\| = 2017 = \|B\|$. Since $|By| \leq \|B\||y|$ for all $y \in \mathbb{R}^n$ and in particular for $y = Ax$, we have

$$\|BA\| = \sup_{|x|=1}|BAx| \leq \sup_{|x|=1}\|B\||Ax| = \|B\|\sup_{|x|=1}|Ax| = \|B\|\|A\| \leq (2017)^2.$$

To conclude, we notice that $(2017)^2 < (2100)^2 = 21^2\cdot 10^4 = 441\cdot 10^4$.

5.2. Volterra equation

Let $(X, \|\cdot\|_X) = \big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$. Since the function k is continuous in both variables, the integral operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s)f(s)\,\mathrm{d}s$$

is well-defined. We claim that for every $n \in \mathbb{N}$ and every $f \in X$ and $t \in [0,1]$,

$$|(T^nf)(t)| \leq \frac{t^n}{n!}\|k\|^n_{C^0([0,1]\times[0,1])}\|f\|_X.$$

We prove the claim by induction. For $n = 1$ we have

$$|(Tf)(t)| \leq \int_0^t |k(t,s)||f(s)|\,\mathrm{d}s \leq t\|k\|_{C^0([0,1]\times[0,1])}\|f\|_X.$$

Suppose the claim is true for some $n \in \mathbb{N}$. Then,

$$\begin{aligned} |(T^{n+1}f)(t)| &\leq \int_0^t |k(t,s)||(T^n f)(s)| \,\mathrm{d}s \\ &\leq \frac{1}{n!}\|k\|_{C^0}^{n+1}\|f\|_X \int_0^t s^n \,\mathrm{d}s = \frac{t^{n+1}}{(n+1)!}\|k\|_{C^0}^{n+1}\|f\|_X \end{aligned}$$

which proves the claim. Since $0 \leq t \leq 1$, the claim implies

$$r_T := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{\|k\|_{C^0}}{(n!)^{\frac{1}{n}}} = 0.$$

From $r_T = 0$ we conclude that the operator $(1 + T) = (1 - (-T))$ is invertible with bounded inverse (Satz 2.2.7). The solution to the Volterra equation $f + Tf = g$ is then given by $f = (1 + T)^{-1}g$.

5.3. Right shift operator

(a) Let $x \in \ell^2$. By definition of S and the ℓ^2-norm $\|Sx\|_{\ell^2} = \|x\|_{\ell^2}$, which implies $\|S\| = 1$. Being linear and bounded, the map S is continuous.

(b) Suppose $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ satisfies $Sx = \lambda x$ for some $\lambda \in \mathbb{R}$. Then

$$(0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3 \ldots).$$

If $\lambda = 0$, then $x = 0$ is immediate. If $\lambda \neq 0$, then $x = 0$ follows via

$$0 = \lambda x_1 \;\Rightarrow\; 0 = x_1 = \lambda x_2 \;\Rightarrow\; 0 = x_2 = \lambda x_3 \;\Rightarrow\; \ldots$$

We conclude that S does not have eigenvalues. The spectral radius of S is

$$r_S = \lim_{n\to\infty} \|S^n\|^{\frac{1}{n}} = 1$$

since $\|S^n\| = 1$ follows for every $n \in \mathbb{N}$ from $\|S^n x\|_{\ell^2} = \|x\|_{\ell^2}$ as in (a).

(c) We define $T\colon \ell^2 \to \ell^2$ to be the left shift map $T\colon (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. Then, $T \circ S = \mathrm{id}$ and $S \circ T \neq \mathrm{id}$. Indeed,

$$(T \circ S)(x_1, x_2, \ldots) = T(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots),$$

$$(S \circ T)(x_1, x_2, \ldots) = S(x_2, x_3, \ldots) = (0, x_2, x_3, \ldots).$$

5.4. Closed subspaces

Since the subspace $V \subset X$ is closed in both statements (a) and (b), the canonical quotient map $\pi\colon X \to X/V$ is continuous (Satz 2.3.1).

(a) $\dim \pi(U) \leq \dim U < \infty$ implies that $\pi(U) \subset X/V$ is closed (Satz 2.1.3). Since π is continuous, $\pi^{-1}(\pi(U)) = U + V \subset X$ is also closed.

(b) Since $\dim \pi(U) \leq \dim(X/V) < \infty$, we can argue the same way as in (a).

5.5. Vanishing boundary values

Let $X = C^0([0,1])$ and $U = C^0_0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}$.

(a) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in U which converges to f in $(X, \|\cdot\|_X)$. Then, since $f_n(0) = 0 = f_n(1)$, we can colclude $f(0) = 0 = f(1)$, i. e. $f \in U$ by passing to the limit $n \to \infty$ in the following inequalities.

$$|f(0)| = |f_n(0) - f(0)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X,$$
$$|f(1)| = |f_n(1) - f(1)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X.$$

(b) Let $u_1, u_2 \in X$ be given by $u_1(t) = 1 - t$ and $u_2(t) = t$. We claim that the equivalence classes $[u_1], [u_2] \in X/U$ form a basis for X/U.

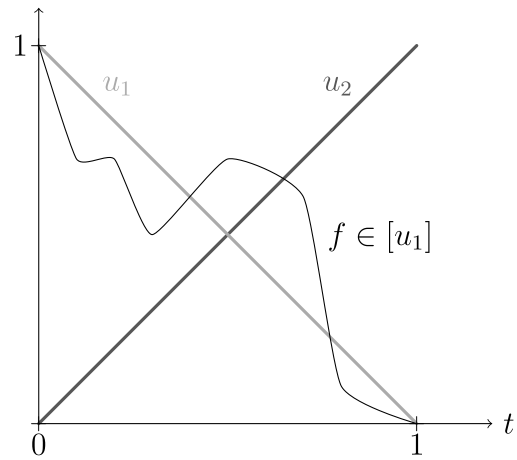

Figure 1: The functions $u_1, u_2 \in X$ and some $f \in [u_1]$.

To prove linear independence, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1[u_1] + \lambda_2[u_2] = 0 \in X/U$ which means $\lambda_1 u_1 + \lambda_2 u_2 \in U$. This implies by definition

$$\lambda_1 = \lambda_1 u_1(0) + \lambda_2 u_2(0) = 0 = \lambda_1 u_1(1) + \lambda_2 u_2(1) = \lambda_2.$$

To show that $[u_1]$ and $[u_2]$ span X/U, let $[h] \in X/U$ with representative $h \in X$. By evaluation at $t = 0$ and $t = 1$, we conclude

$$\Big(t \mapsto h(t) - h(0)u_1(t) - h(1)u_2(t)\Big) \in U.$$

This implies $[h] = h(0)[u_1] + h(1)[u_2]$ in X/U which proves the claim.

Remark. The components of $[h]$ in this basis are unique since every representative $\tilde{h} \in [h]$ must have the same boundary values $\tilde{h}(0) = h(0)$ and $\tilde{h}(1) = h(1)$.

5.6. Topological complement

(a) Suppose, $U \subset X$ is topologically complemented by $V \subset X$. Then, $I\colon U \times V \to X$ with $(u, v) \mapsto u + v$ is an continuous isomorphism with continuous inverse. We define

$$P_1\colon U \times V \to U \times V, \qquad P := I \circ P_1 \circ I^{-1}\colon X \to X.$$
$$(u, v) \mapsto (u, 0)$$

P_1 is linear, bounded since $\|P_1(u, v)\|_{U\times V} = \|u\|_U \le \|(u, v)\|_{U\times V}$ and hence continuous. As composition of linear continuous maps, P is linear and continuous. Moreover,

$$P \circ P = (I \circ P_1 \circ I^{-1}) \circ (I \circ P_1 \circ I^{-1}) = I \circ P_1 \circ P_1 \circ I^{-1} = I \circ P_1 \circ I^{-1} = P,$$
$$P(X) = I(U \times \{0\}) = U.$$

Conversely, suppose $U \subset X$ allows a continuous linear map $P\colon X \to X$ with $P\circ P = P$ and $P(X) = U$. Let $V := \ker(P)$. Then

$$P \circ (1 - P) = P - P = 0 \qquad \Rightarrow (1 - P)(X) \subseteq \ker(P) = V. \tag{1}$$

In fact, $(1 - P)(X) = V$ since given $v \in V$ we have $v = (1 - P)v$. Analogously,

$$(1 - P) \circ P = P - P = 0 \qquad \Rightarrow\ U = P(X) \subseteq \ker(1 - P). \tag{2}$$

In fact, $U = \ker(1 - P)$ since $x - Px = 0$ implies $x = Px \in U$. The claim is, that

$$I\colon U \times V \to X$$
$$(u, v) \mapsto u + v$$

is continuous and has a continuous inverse. Continuity of I follows directly from

$$\|I(u, v)\|_X = \|u + v\|_X \le \|u\|_X + \|v\|_X = \|(u, v)\|_{U\times V}.$$

By the assumptions on P, especially (1), the map

$$\Phi\colon X \to U \times V$$
$$x \mapsto \bigl(Px, (1 - P)x\bigr)$$

is well-defined and continuous. Since $Pu = u$ for all $u \in U$ by (2) we have

$$(\Phi \circ I)(u, v) = \Phi(u + v) = \Bigl(Pu + Pv, u - Pu + v - Pv\Bigr) = (u, v).$$
$$(I \circ \Phi)(x) = I(Px, (1 - P)x) = Px + (1 - P)x = x,$$

which implies that Φ is inverse to I. Consequently, U is topologically complemented.

(b) If $U \subset X$ is topologically complemented, then (a) implies existence of a continuous map $P\colon X \to X$ with $\ker(1 - P) = U$. Thus, U must be closed as the kernel of the continuous map $1 - P$.

Remark. If X is *not* isomorphic to a Hilbert space, then X has closed subspaces which are *not* topologically complemented [Lindenstrauss & Tzafriri. *On the complemented subspaces problem.* (1971)]. An example is $c_0 \subset \ell^\infty$ but this is not easy to prove.

5.7. Continuity of bilinear maps

(a) Let $((x_k, y_k))_{k\in\mathbb{N}}$ be a sequence in $X\times Y$ converging to (x,y) in $(X\times Y, \|\cdot\|_{X\times Y})$. By definition,

$$\|x_k - x\|_X + \|y_k - y\|_Y = \|(x_k - x, y_k - y)\|_{X\times Y} = \|(x_k, y_k) - (x,y)\|_{X\times Y}$$

which yields convergence $x_k \to x$ in X and $y_k \to y$ in Y. Since $B\colon X\times Y \to Z$ is bilinear, we have

$$\begin{aligned}\|B(x_k, y_k) - B(x,y)\|_Z &= \|B(x_k,y_k) - B(x,y_k) + B(x,y_k) - B(x,y)\|_Z \\ &= \|B(x_k - x, y_k) - B(x, y_k - y)\|_Z \\ &\leq \|B(x_k - x, y_k)\|_Z + \|B(x, y_k - y)\|_Z.\end{aligned}$$

Using the assumption $\|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y$ and the fact, that convergence of $(y_k)_{k\in\mathbb{N}}$ in $(Y, \|\cdot\|_Y)$ implies that $\|y_k\|_Y$ is bounded uniformly for all $k\in\mathbb{N}$, we conclude

$$\|B(x_k, y_k) - B(x,y)\|_Z \leq C\|x - x_k\|_X\|y_k\|_Y + C\|x\|_X\|y - y_k\|_Y \xrightarrow{k\to\infty} 0.$$

(b) Let $B_1^Y \subset Y$ be the unit ball around the origin in $(Y, \|\cdot\|_Y)$. For every $x\in X$ we have by assumption

$$\sup_{y'\in B_1^Y}\|B(x,y')\|_Z \leq \sup_{y'\in B_1^Y}\|y'\|_Y\|B(x,\cdot)\|_{L(Y,Z)} \leq \|B(x,\cdot)\|_{L(Y,Z)} < \infty,$$

which means that the maps $(B(\cdot, y'))_{y'\in B_1^Y} \in L(X,Z)$ are pointwise bounded. Since X is assumed to be complete, the Theorem of Banach-Steinhaus implies that $(B(\cdot,y'))_{y'\in B_1^Y} \in L(X,Z)$ are uniformly bounded, i. e.

$$C := \sup_{y'\in B_1^Y}\|B(\cdot, y')\|_{L(X,Z)} < \infty.$$

From that we conclude

$$\begin{aligned}\|B(x,y)\|_Z &= \|y\|_Y\left\|B\Big(x, \frac{y}{\|y\|_Y}\Big)\right\|_Z \\ &\leq \|y\|_Y\|x\|_X\left\|B\Big(\cdot, \frac{y}{\|y\|_Y}\Big)\right\|_{L(X,Z)} \leq C\|x\|_X\|y\|_Y.\end{aligned}$$