6.1. Graph of bounded functions

Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function with graph $\Gamma\subset\mathbb{R}^2$. Let $((x_n,y_n))_{n\in\mathbb{N}}$ be a sequence in Γ converging to some $(x,y)\in\mathbb{R}^2$. Then, continuity of f implies

$$f(x)=f\Big(\lim_{n\to\infty}x_n\Big)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}y_n=y$$

which means that $(x,y)\in\Gamma$. Therefore, Γ is closed.

Conversely, let $f\colon\mathbb{R}\to\mathbb{R}$ be a bounded function with closed graph Γ. Given $x\in\mathbb{R}$, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}$ with limit x. Let $y_n=f(x_n)$. Since f is bounded, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded and therefore has a convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$. Let $y\in\mathbb{R}$ be its limit. Since Γ is closed, $(x,y)=\lim_{k\to\infty}(x_{n_k},y_{n_k})\in\Gamma$ which means that $f(x)=y$. Therefore, $y=f(x)$ the unique limit of any convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is bounded with only one accumulation point $y\in\mathbb{R}$ we have $y_n\to y$ as $n\to\infty$. This implies continuity of f at x by virtue of

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}y_n=y=f(x)=f\Big(\lim_{n\to\infty}x_n\Big).$$

If f is *not* assumed to be bounded, we can construct the following counterexample.

$$f(x)=\begin{cases}\frac{1}{x} & \text{for } x>0,\\ 0 & \text{for } x\leq 0.\end{cases}$$

Since both restrictions $f|_{]0,\infty[}$ and $f|_{]-\infty,0]}$ are continuous, both connected components of the graph of f are closed but f is not continuous at $x=0$.

6.2. An implication of Hellinger-Töplitz (coercive operators)

If $Ax=0$, then the assumtion implies $\lambda\|x\|^2\leq(Ax,x)=0$. Since $\lambda>0$, we have $x=0$ which proves that the linear map A is injective.

Let $W_A:=A(H)$ be the range of A. To prove that A is surjective, let $x\in W_A^\perp$. Then

$$0=(Ax,x)\geq\lambda\|x\|^2$$

which implies $x=0$. Therefore, $W_A^\perp=\{0\}$ and $\overline{W_A}=(W_A^\perp)^\perp=H$. Surjectivity of A follows if we show that W_A is closed in H because then, $W_A=\overline{W_A}=H$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in W_A converging to some $y\in H$. Let $x_n\in H$ such that $Ax_n=y_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in H, because for every $n,m\in\mathbb{N}$ we have

$$\begin{aligned}\lambda\|x_n-x_m\|^2&\leq(Ax_n-Ax_m,x_n-x_m)=(y_n-y_m,x_n-x_m)\\&\leq\|y_n-y_m\|\|x_n-x_m\|\end{aligned}$$

and $(y_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence by assumption. Hence there exists $x=\lim_{n\to\infty}x_n$. The Hellinger–Töplitz theorem (Beispiel 3.3.2) implies that A is continuous. Therefore, $Ax=y$ which implies $y\in W_A$ and proves that W_A is closed in H.

We have shown that A is a continuous, bijective linear operator. The Inverse Mapping Theorem already implies that A has a continuous inverse. What remains to show is the estimate $\|A^{-1}\| \leq \frac{1}{\lambda}$ which follows from the assumption since for every $y \in H$

$$\begin{aligned} &\|A^{-1}y\|^2 \leq \frac{1}{\lambda}\Big(AA^{-1}y, A^{-1}y\Big) \leq \frac{1}{\lambda}\|y\|\|A^{-1}y\| \\ \Rightarrow\ &\|A^{-1}y\| \leq \frac{\|y\|}{\lambda}. \end{aligned}$$

6.3. Derivative operator

Let $f \in D_{\overline{A}}$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A with

$$\|f_n - f\|_{L^2([0,1])} \to 0, \qquad \|Af_n - \overline{A}f\|_{L^2([0,1])} \to 0$$

as $n \to \infty$. By definition, $f_n \in C_c^\infty(]0,1[) \subset L^2([0,1])$ has a representative which (after extension to $t = 0$ and $t = 1$) is a smooth function $\tilde{f}_n\colon [0,1] \to \mathbb{R}$ with $\tilde{f}_n(0) = 0 = \tilde{f}_n(1)$. But L^2-convergence alone is *not* enough to conclude the same for f. Instead, we observe, that $\tilde{f}_n$ is a primitive of $A\tilde{f}_n = \tilde{f}_n'$. We hope that f arises as a primitive of $\overline{A}f \in L^2([0,1])$. Therefore, we consider the function $g\colon [0,1] \to \mathbb{R}$ given by

$$g(t) := \int_0^t \overline{A}f \,\mathrm{d}x.$$

We apply Hölder's inequality to estimate

$$\begin{aligned} \Big|\tilde{f}_n(t) - g(t)\Big| &= \left|\int_0^t \Big(\tilde{f}_n' - \overline{A}f\Big)\,\mathrm{d}x\right| \\ &\leq \int_0^t \Big|A\tilde{f}_n - \overline{A}f\Big|\,\mathrm{d}x \\ &\leq \left(\int_0^t 1\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^t \Big|A\tilde{f}_n - \overline{A}f\Big|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \Big\|A\tilde{f}_n - \overline{A}f\Big\|_{L^2([0,1])} \xrightarrow{n\to\infty} 0. \end{aligned}$$

Taking the supremum over $t \in [0,1]$, we see that $(\tilde{f}_n)_{n\in\mathbb{N}}$ converges *uniformly* to g. Since uniform convergence implies L^2-convergence, g must coincide with the L^2-limit f of $(\tilde{f}_n)_{n\in\mathbb{N}}$. As a uniform limit of continuous functions, $g = f$ is continuous. Finally, uniform convergence implies pointwise convergence, in particular

$$f(0) = \lim_{n\to\infty} \tilde{f}_n(0) = 0, \qquad f(1) = \lim_{n\to\infty} \tilde{f}_n(1) = 0.$$

6.4. Closed range

Given a linear operator $A\colon D_A \subset X \to Y$ with closed graph the claim is equivalence of

(i) A is injective and its range $W_A := A(D_A)$ is closed in $(Y, \|\cdot\|_Y)$.

(ii) $\exists C > 0 \quad \forall x \in D_A : \quad \|x\|_X \leq C\|Ax\|_Y$

Proof. "(i) $\Rightarrow$ (ii)" As a closed subspace of a complete space, $(W_A, \|\cdot\|_Y)$ is complete. Since $A\colon D_A \subset X \to W_A$ is bijective with closed graph and since X, W_A are Banach spaces and we may apply the Inverse Mapping Theorem to obtain a continuous inverse $A^{-1}\colon W_A \to D_A$. In particular, $\|A^{-1}\| =: C$ is finite and for every $x \in D_A$ we have

$$\|x\|_X = \|A^{-1}Ax\|_X \leq \|A^{-1}\|\|Ax\|_Y = C\|Ax\|_Y.$$

"(ii) $\Rightarrow$ (i)" Let $x \in D_A$ with $Ax = 0$. Then, (ii) implies $\|x\|_X \leq 0$, hence $x = 0$. This implies that the linear map A is injective.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in W_A converging to some $y \in Y$. By definition of W_A there exist $x_n \in D_A$ such that $Ax_n = y_n$. For every $n, m \in \mathbb{N}$, assumption (ii) implies

$$\|x_n - x_m\|_X \leq C\|Ax_n - Ax_m\|_Y = C\|y_n - y_m\|_Y.$$

From $(y_n)_{n\in\mathbb{N}}$ being Cauchy in $(Y, \|\cdot\|_Y)$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|\cdot\|_X)$. Since $(X, \|\cdot\|_X)$ is complete, there exists $X \ni x = \lim_{n\to\infty} x_n$. Since the graph of A is assumed to be closed, $x \in D_A$ and $Ax = y$. Therefore, $y \in W_A$ and we conclude that W_A is a closed subspace of Y. □

6.5. Graph norm

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in $(D_A, \|\cdot\|_{\Gamma_A})$. The definition of graph norm implies that then, $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|\cdot\|_X)$ and $(Ax_n)_{n\in\mathbb{N}}$ is Cauchy in $(Y, \|\cdot\|_Y)$. Since both spaces are complete, there exist $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|Ax_n - y\|_Y \to 0$ as $n \to \infty$. Since the graph of A is closed, we have $x \in D_A$ with $Ax = y$. Therefore, $\|x - x_n\|_{\Gamma_A} \to 0$ which proves that $(D_A, \|\cdot\|_{\Gamma_A})$ is complete.

(b) Let Γ_1 and Γ_2 be the graphs of T_1 and T_2 respectively. Since T_1 and T_2 have closed graphs by assumption, (a) implies that $(D_1, \|\cdot\|_{\Gamma_1})$ and $(D_2, \|\cdot\|_{\Gamma_2})$ are Banach spaces. Since $D_1 \subset D_2$, we can consider the identity map $\mathrm{Id}\colon (D_1, \|\cdot\|_{\Gamma_1}) \to (D_2, \|\cdot\|_{\Gamma_2})$ and claim that its graph is closed. Indeed, assume that $x_n \to x$ in $(D_1, \|\cdot\|_{\Gamma_1})$ and $\mathrm{Id}(x_n) = x_n \to y$ in $(D_2, \|\cdot\|_{\Gamma_2})$. Then, the definition of graph norm implies that both, $\|x_n - x\|_{X_0} \to 0$ and $\|x_n - y\|_{X_0} \to 0$ as $n \to \infty$ which implies $x = y$ and proves the claim. The closed graph theorem implies that Id is continuous, which means

$$\exists C > 0 \quad \forall x \in D_1 : \quad \|x\|_{\Gamma_2} \leq C\|x\|_{\Gamma_1}.$$

By definition, this implies $\|T_2x\|_{X_2} \leq C\Big(\|T_1x\|_{X_1} + \|x\|_{X_0}\Big) - \|x\|_{X_0}$.

6.6. Closed sum

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D_A. Then, by triangle inequality and the assumption,

$$\begin{aligned}\|A(x_n - x_m)\|_Y - \|(A+B)(x_n - x_m)\|_Y &\leq \|B(x_n - x_m)\|_Y \\ &\leq a\|A(x_n - x_m)\|_Y + b\|x_n - x_m\|_X.\end{aligned}$$

which implies the estimate

$$(1-a)\|A(x_n - x_m)\|_Y \leq \|(A+B)(x_n - x_m)\|_Y + b\|x_n - x_m\|_X. \qquad (\dagger)$$

Assume that $x_n \to x$ in X and $(A+B)x_n \to y$ in Y. The claim is $(A+B)x = y$. Since $a < 1$, estimate $(\dagger)$ implies that $(Ax_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $(Y, \|\cdot\|_Y)$ and therefore convergent to some $\tilde{y} \in Y$. Since the graph of A is a closed by assumption, we have $x \in D_A$ with $Ax = \tilde{y}$. Therefore, we may conclude

$$\|B(x - x_n)\|_Y \leq a\|A(x - x_n)\| + b\|x - x_n\| \xrightarrow{n\to\infty} 0$$

which implies $Bx_n \to Bx$ in Y and thus,

$$y = \lim_{n\to\infty}(A+B)x_n = \lim_{n\to\infty} Ax_n + \lim_{n\to\infty} Bx_n = Ax + Bx = (A+B)x.$$

6.7. Closable inverse

Since the closure $\overline{A}$ is assumed to be injective, A is injective and therefore has an inverse $A^{-1}\colon W_A \to D_A$, where $W_A := A(D_A)$ denotes the range of A. Defining

$$\begin{aligned}\chi\colon X \times Y &\to Y \times X \\ (x,y) &\mapsto (y,x)\end{aligned}$$

we observe that the graph $\Gamma_{A^{-1}}$ of A^{-1} is given by

$$\Gamma_{A^{-1}} := \{(y,x) \in Y \times X \mid y \in W_A,\ x = A^{-1}y\} = \chi(\Gamma_A).$$

Since χ is an isomorphism of normed spaces, we have

$$\overline{\Gamma_{A^{-1}}} = \overline{\chi(\Gamma_A)} = \chi(\overline{\Gamma_A}) = \chi(\Gamma_{\overline{A}}) = \Gamma_{(\overline{A})^{-1}}$$

This proves that $\overline{\Gamma_{A^{-1}}}$ is the graph of the linear operator $(\overline{A})^{-1}$ (which is well-defined, since $\overline{A}$ is injective). Therefore, A^{-1} is closable as claimed and

$$\Gamma_{\overline{A^{-1}}} = \overline{\Gamma_{A^{-1}}} = \Gamma_{(\overline{A})^{-1}} \quad \Rightarrow \ \overline{A^{-1}} = (\overline{A})^{-1}.$$