7.1. Derivative operator on different spaces

(a) The operator $\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset C^0([0,1]) \to C^0([0,1])$ is not bounded. A counterexample are the functions $f_n \in C^1([0,1])$ for $n \in \mathbb{N}$ given by $f_n(t) = t^n$. Indeed,

$$\left.\begin{aligned} \|f_n\|_{C^0([0,1])} &= \max_{t\in[0,1]} t^n = 1, \\ \|\tfrac{\mathrm{d}}{\mathrm{d}t} f_n\|_{C^0([0,1])} &= \max_{t\in[0,1]} nt^{n-1} = n, \end{aligned}\right. \qquad \Rightarrow \quad \frac{\|\frac{\mathrm{d}}{\mathrm{d}t} f_n\|_{C^0([0,1])}}{\|f_n\|_{C^0([0,1])}} = n.$$

To check, whether the operator is closable, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of functions $u_k \in C^1([0,1])$ such that $\|u_k\|_{C^0([0,1])} \to 0$ as $k \to \infty$. Suppose, $v \in C^0([0,1])$ is a limit of $v_k := \frac{\mathrm{d}}{\mathrm{d}t} u_k$ in the sense that $\|v - v_k\|_{C^0([0,1])} \to 0$. Does $v = 0$ follow? Yes, in fact, for any $\varphi \in C_c^\infty(]0,1[)$, integration by parts yields (the boundary terms vanish due to $\varphi(0) = 0 = \varphi(1)$)

$$\left|\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t\right| = \left|-\int_0^1 u_k(t)\varphi'(t)\,\mathrm{d}t\right| \leq \left(\int_0^1 |\varphi'(t)|\,\mathrm{d}t\right)\|u_k\|_{C^0([0,1])} \xrightarrow{k\to\infty} 0.$$

Since $\|v - v_k\|_{C^0([0,1])} \to 0$ implies

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t = \lim_{k\to\infty}\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t = 0$$

and since $\varphi \in C_c^\infty(]0,1[)$ is arbitrary, we have $v(t) = 0$ for almost every $t \in [0,1]$. As v is continuous, this implies $v \equiv 0$ on $[0,1]$. Therefore, the operator is closable.

(b) The operator $\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset L^2([0,1]) \to L^2([0,1])$ is not bounded. A counterexample are the functions $g_n \in C^1([0,1])$ for $n \in \mathbb{N}$ given by $g_n(t) = \mathrm{e}^{nt}$. Indeed,

$$\left.\begin{aligned} \|g_n\|_{L^2([0,1])} &= \left(\int_0^1 \mathrm{e}^{2nt}\,\mathrm{d}t\right)^{\frac12} = \frac{1}{\sqrt{2n}}, \\ \|\tfrac{\mathrm{d}}{\mathrm{d}t} g_n\|_{L^2([0,1])} &= \left(\int_0^1 (n\mathrm{e}^{nt})^2\,\mathrm{d}t\right)^{\frac12} = \frac{n}{\sqrt{2n}}, \end{aligned}\right. \qquad \Rightarrow \quad \frac{\|\frac{\mathrm{d}}{\mathrm{d}t} g_n\|_{L^2([0,1])}}{\|g_n\|_{L^2([0,1])}} = n.$$

To check, whether the operator is closable, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of functions $u_k \in C^1([0,1])$ such that $\|u_k\|_{L^2([0,1])} \to 0$ as $k \to \infty$. Suppose, $v \in L^2([0,1])$ is a limit of $v_k := \frac{\mathrm{d}}{\mathrm{d}t} u_k$ in the sense that $\|v - v_k\|_{L^2([0,1])} \to 0$. Does $v = 0$ follow? Yes, in fact, for any $\varphi \in C_c^\infty(]0,1[)$ using Hölder's inequality, we have

$$\left|\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t\right| = \left|-\int_0^1 u_k(t)\varphi'(t)\,\mathrm{d}t\right| \leq \|u_k\|_{L^2([0,1])}\|\varphi'\|_{L^2([0,1])} \xrightarrow{n\to\infty} 0.$$

Since $\|v - v_k\|_{L^2([0,1])} \to 0$ implies (for instance by continuity of the L^2-scalar product)

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t = \lim_{k\to\infty}\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t = 0$$

and since $\varphi \in C_c^\infty(]0,1[)$ is arbitrary, we have $v = 0$ in $L^2([0,1])$. (For that we do not care about the values at $t = 0$ or $t = 1$.) Therefore, the operator is closable.

7.2. Complementing subspaces of finite dimension or codimension

(a) Let $e_1, \ldots, e_n$ be a basis of the given finite-dimensional subspace $U \subset X$. For $i \in \{1, \ldots, n\}$, we define the linear functionals $f_i \colon U \to \mathbb{R}$ by

$$f_i(e_j) = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{else.} \end{cases}$$

Recall that by linearity, it suffices to define the functionals on a basis of U. Since U is finite-dimensional, $f_i \in L(U; \mathbb{R})$. From the Hahn-Banach Theorem follows (Satz 4.1.3) that there exist extensions $F_i \in L(X; \mathbb{R})$ with $\|F_i\| = \|f_i\|$. We define

$$\begin{aligned} P \colon X &\to X \\ x &\mapsto \sum_{i=1}^{n} F_i(x)\, e_i. \end{aligned}$$

Then, P is linear and also continuous, since

$$\|Px\|_X \leq \Big(\sum_{i=1}^{n} \|F_i\| \|e_i\|_X\Big) \|x\|_X.$$

By construction, $P(X) \subset \operatorname{span}\{e_1, \ldots, e_n\} = U$. By definition of f_i and F_i we have $P(e_i) = e_i$ for every $i \in \{1, \ldots, n\}$. Therefore, $P(X) = U$. Finally, for every $x \in X$,

$$(P \circ P)(x) = P\Big(\sum_{i=1}^{n} F_i(x)\, e_i\Big) = \sum_{i=1}^{n} F_i(x)\, P(e_i) = \sum_{i=1}^{n} F_i(x)\, e_i = P(x).$$

From Problem 5.6 (a) then follows that U is topologically complemented.

(b) Recall that the quotient space X/U consists of equivalence classes which we denote by $[x]$ and comes with a canonical quotient map $\pi \colon X \to X/U$. Since $\dim(X/U) = m < \infty$ we can choose a basis $[e_1], \ldots, [e_m]$ for X/U along with a representative $e_i \in X$ for each element $[e_i]$ of the basis. As in (a) we define linear functionals $f_i \colon X/U \to \mathbb{R}$ for $i \in \{1, \ldots, m\}$ by $f_i([e_j]) = \delta_{ij}$. Now, we just set $F_i := f_i \circ \pi \colon X \to \mathbb{R}$ in order to define

$$\begin{aligned} P \colon X &\to X \\ x &\mapsto \sum_{i=1}^{n} F_i(x)\, e_i. \end{aligned}$$

Since $F_i(e_j) = f_i(\pi(e_j)) = f_i([e_j]) = \delta_{ij}$ we have $P \circ P = P$ as in (a). Since $[e_1], \ldots, [e_m]$ is a basis for X/U, the representatives $e_1, \ldots, e_m$ must be linearly independent in X. Therefore, $P(x) = 0$ implies $F_i(x) = f_i([x]) = 0$ for every $i \in \{1, \ldots, n\}$ which in turn implies $[x] = [0]$ or $x \in U$. Conversely, $x \in U$ implies $\pi(x) = [0]$ and $P(x) = 0$. Thus we have shown $\ker(P) = U$. As in Problem 5.6 (a), we conclude that $(1 - P)$ is a continuous projection onto U which implies that U is topologically complemented.

7.3. Dense kernel

Given a normed space $(X, \|\cdot\|_X)$, the claim is that the linear map $0 \not\equiv f \colon X \to \mathbb{R}$ is not continuous if and only if $\ker(f)$ is dense in X.

"$\Rightarrow$" Suppose, f is not continuous. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X, which can be normed to $\|x_k\|_X = 1$ by linearity of f, such that $|f(x_k)| \to \infty$ as $k \to \infty$. Without loss of generality, we can assume $f(x_k) \neq 0$ for every $k \in \mathbb{N}$. The goal is to approximate any $z \in X$ by elements $y_k \in \ker(f)$. For each $k \in \mathbb{N}$ we define

$$y_k := z - \frac{f(z)}{f(x_k)} x_k, \qquad \Rightarrow\ f(y_k) = f(z) - \frac{f(z)}{f(x_k)} f(x_k) = 0 \qquad \Rightarrow\ y_k \in \ker(f).$$

Indeed, the sequence $(y_k)_{k\in\mathbb{N}}$ approximates z in X because

$$\|z - y_k\|_X = \left|\frac{f(z)}{f(x_k)}\right| \|x_k\|_X = \frac{|f(z)|}{|f(x_k)|} \xrightarrow{k\to\infty} 0$$

and we have shown that $\ker(f)$ is dense in X.

"$\Leftarrow$" Conversely, we assume $\overline{\ker(f)} = X$ and claim that f is not continuous. Since we assume $f \not\equiv 0$ there exists $x \in X$ with $f(x) \neq 0$. Since the kernel is dense, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\ker(f)$ with $\|x_k - x\|_X \to 0$ as $k \to \infty$. But this violates continuity: $\lim\limits_{k\to\infty} f(x_k) = 0 \neq f(x)$.

7.4. Attaining the distance from the kernel

Given a normed space $(X, \|\cdot\|_X)$, a continuous linear functional $\varphi \colon X \to \mathbb{R}$ with kernel $N := \ker(\varphi) \subsetneq X$ and a point $x_0 \in X \setminus N$, the claim is equivalence of

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N) =: d$,

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

The first isomorphism theorem states that the quotient space X/N is isomorphic to the image of φ. Since $\varphi(x_0) \neq 0$ and $\varphi(\lambda x_0) = \lambda\varphi(x_0)$ for every $\lambda \in \mathbb{R}$, the image of φ is $\mathbb{R}$ which means that X/N is one-dimensional. Therefore, every element $[x] \in X/N$ is of the form $[x] = t[x_0]$ with uniquely determined $t \in \mathbb{R}$. This means that every element $x \in X$ is of the form $x = tx_0 + y$ with uniquely determined $t \in \mathbb{R}$ and $y \in N$.

"(i) $\Rightarrow$ (ii)" Let $x_1 = \frac{1}{d}(x_0 - y_0)$. Then $\|x_1\|_X = 1$. We hope that $|\varphi(x_1)| = \|\varphi\|$ and start estimating

$$|\varphi(x_1)| = \frac{|\varphi(x_1)|}{\|x_1\|_X} \leq \sup_{x\in X} \frac{|\varphi(x)|}{\|x\|_X} = \|\varphi\|.$$

Given any $x \in X$, let $t \in \mathbb{R}$ and $y \in N$ be as above. If $t = 0$, then $\varphi(x) = 0$ is not interesting. Therefore, we assume $t \neq 0$ and observe

$$|\varphi(x)| = |\varphi(tx_0 + y)| = |\varphi(td\,x_1 + t\,y_0 + y)| = |t|d\,|\varphi(x_1)|,$$
$$\|x\|_X = \|tx_0 + y\|_X = |t|\|x_0 + \tfrac{1}{t}y\|_X \geq |t| \inf_{\tilde{y}\in N}\|x_0 - \tilde{y}\|_X = |t|d.$$

This implies that

$$|\varphi(x_1)| = \frac{|\varphi(x_1)|}{\|x_1\|_X} \leq \sup_{x\in X}\frac{|\varphi(x)|}{\|x\|_X} \leq \frac{|t|d\,|\varphi(x_1)|}{|t|d} = |\varphi(x_1)|.$$

Thus, the inequalities above are in fact identities and we conclude $|\varphi(x_1)| = \|\varphi\|$.

"(ii) $\Rightarrow$ (i)" As above, we have $x_1 = tx_0 + y_1$ for uniquely determined $t \in \mathbb{R}$ and $y_1 \in N$. In fact, $t \neq 0$ since $|\varphi(x_1)| = \|\varphi\| \neq 0$. Therefore, $x_0 = \frac{1}{t}(x_1 - y_1)$ and

$$\|x_0 + \tfrac{1}{t}y_1\|_X = \|\tfrac{1}{t}x_1\|_X = \frac{1}{|t|}.$$

In the following, we use the fact that any $z \in X$ satisfies the estimate

$$\|\varphi\| \geq \frac{|\varphi(z)|}{\|z\|_X} \quad \Rightarrow \|z\|_X \geq \frac{|\varphi(z)|}{\|\varphi\|}.$$

Given any $y \in N$, we have

$$\begin{aligned}\|x_0 - y\|_X &= \|\tfrac{1}{t}x_1 - \tfrac{1}{t}y_1 - y\|_X \\ &\geq \frac{|\varphi(\frac{1}{t}x_1 - \frac{1}{t}y_1 - y)|}{\|\varphi\|} = \frac{|\varphi(x_1)|}{|t|\|\varphi\|} = \frac{1}{|t|} = \|x_0 + \tfrac{1}{t}y_1\|_X.\end{aligned}$$

Since $y_0 := -\frac{1}{t}y_1 \in N$ we conclude that y_0 attains $\operatorname{dist}(x_0, N)$.

7.5. Not attaining the distance from the kernel

From problem 4.5 (a) we know that $\varphi\colon X \to \mathbb{R}$ is a continuous linear functioal. Therefore $N = \ker(\varphi)$ is a closed subspace of X. From problem 4.5 (b) we know that $\|\varphi\| = 2$. From problem 4.5 (c) we know that there does not exist any $x_1 \in X$ with $\|x_1\|_X = 1$ and $|\varphi(x_1)| = 2 = \|\varphi\|$. From problem 7.4 we know that this is equivalent to the statement that there does not exist any $y_0 \in N$ with $\|x_0 - y_0\| = \operatorname{dist}(x_0, N)$.

7.6. Unique extension of functionals on Hilbert spaces

As a continuous linear operator $f\colon Y\subset H\to\mathbb{R}$ is closable. Let $\overline{f}$ be its closure. We claim that $D(\overline{f})=\overline{Y}$. A priori, we only know $D(\overline{f})\subset\overline{D(f)}=\overline{Y}$. Therefore, we consider $y\in\overline{Y}$ together with a sequence $(y_k)_{k\in\mathbb{N}}$ in Y such that $\|y-y_k\|_H\to 0$ as $k\to\infty$. From

$$|f(y_n)-f(y_m)|\leq\|f\|\|y_n-y_m\|_H,$$

we conclude that $(f(y_k))_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\mathbb{R}$. Thus, there exists $z\in\mathbb{R}$ such that $f(y_k)\to z$ as $k\to\infty$. This means that (y,z) is in the closure of the graph of f and we conclude $y\in D(\overline{f})$. Moreover, by continuity of the norm,

$$|\overline{f}(y)|=\lim_{k\to\infty}|f(y_k)|\leq\lim_{k\to\infty}\|f\|\|y_k\|_H=\|f\|\|y\|_H\qquad\Rightarrow\ \|\overline{f}\|=\|f\|.$$

The argument above shows that in order to extend $f\colon Y\to\mathbb{R}$, we can first uniquely extend to $\overline{f}\colon\overline{Y}\to\mathbb{R}$ without changing the norm and then extend to $F\colon H\to\mathbb{R}$ with the advantage that we can now work with the *closed* subspace $\overline{Y}$. In fact, $(\overline{Y},(\cdot,\cdot)_H)$ is a Hilbert space! This allows us to apply the Riesz representation theorem: There exists a unique $h\in\overline{Y}$ such that for all $y\in\overline{Y}$

$$\overline{f}(y)=(y,h)_H.$$

This suggests the extension

$$\begin{aligned}F\colon H&\to\mathbb{R}\\ x&\mapsto(x,h)_H\end{aligned}$$

which satisfies $\|F\|=\|h\|_H=\|\overline{f}\|=\|f\|$. Is this extension unique? If $\tilde{F}\colon H\to\mathbb{R}$ is another extension of $\overline{f}$ with $\|\tilde{F}\|=\|\overline{f}\|$ then it must be also of the form $\tilde{F}(x)=(x,\tilde{h})_H$ for some $\tilde{h}\in H$ by the Riesz representation theorem. Is $h=\tilde{h}$? Since $\tilde{F}|_{\overline{Y}}=\overline{f}=F|_{\overline{Y}}$, we have

$$\forall y\in\overline{Y}:\quad 0=F(y)-\tilde{F}(y)=(y,h)_H-(y,\tilde{h})_H=(y,h-\tilde{h})_H$$

which implies $h-\tilde{h}\in\overline{Y}^{\perp}$. Since $h\in\overline{Y}$ and $\|h\|_H=\|F\|=\|\tilde{F}\|=\|\tilde{h}\|$, we have

$$\|h\|_H^2=\|\tilde{h}\|_H^2=\|\tilde{h}-h+h\|_H^2=\|\tilde{h}-h\|_H^2+\|h\|_H^2,$$

where we used $(\tilde{h}-h)\perp h$. This implies $\|\tilde{h}-h\|_H^2=0$. Therefore, F is the unique extension of f with $\|F\|=\|f\|$.

7.7. Distance from convex sets in Hilbert spaces

Without loss of generality, we can assume $x = 0$. Otherwise we apply the translation $y \mapsto y - x$ which is an isometry to the entire space H.

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the convex set $Q \subset H$ with $\|x_n\| \to d = \operatorname{dist}(0, Q)$ as $n \to \infty$. By convexity of Q, the implications

$$x_n, x_m \in Q \qquad \Rightarrow \frac{x_n + x_m}{2} \in Q \qquad \Rightarrow \left\|\frac{x_n + x_m}{2}\right\|_H \geq \operatorname{dist}(0, Q) = d$$

hold for all $n, m \in \mathbb{N}$. The parallelogram identity "$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$" which is true in Hilbert spaces yields

$$\begin{aligned}\|x_n - x_m\|_H^2 &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - \|x_n + x_m\|_H^2 \\ &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4\left\|\frac{x_n + x_m}{2}\right\|_H^2 \leq 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4d^2.\end{aligned}$$

From $2\|x_n\|_H^2 \to 2d^2$ as $n \to \infty$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in H.

(b) Now we assume that the convex set $Q \subset H$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Q with $\|x_n\|_H \to d = \operatorname{dist}(0, Q)$ as $n \to \infty$. According to (a), it must be a Cauchy-sequence. Since H is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $y \in H$. In fact, $y \in Q$ since Q is closed.

Suppose there is another point $\tilde{y} \in Q$ with $\|\tilde{y}\| = d$. Then, again by convexity and the parallelogram identity,

$$d^2 \leq \left\|\frac{y + \tilde{y}}{2}\right\|_H^2 \leq \left\|\frac{y + \tilde{y}}{2}\right\|_H^2 + \left\|\frac{y - \tilde{y}}{2}\right\|_H^2 = \frac{1}{2}\|y\|_H^2 + \frac{1}{2}\|\tilde{y}\|_H^2 = d^2$$

and we conclude that all the inequalities are in fact identities which implies

$$\left\|\frac{y - \tilde{y}}{2}\right\|_H^2 = 0.$$

Thus, $y = \tilde{y}$ and we have proven existence and uniqueness of $y \in Q$ with $\|y\|_H = d$.