9.1. Minkowski functional

Given the normed space $(X, \|\cdot\|_X)$, the non-trivial, open, convex subset $Q \subset X$ and the Minkowski functional

$$\begin{aligned} p\colon X &\to \mathbb{R} \\ x &\mapsto \inf\{\lambda > 0 \mid \tfrac{1}{\lambda}x \in Q\} \end{aligned}$$

we define the set

$$\Upsilon := \{f \in X^* \mid \forall x \in X: \ f(x) \leq p(x)\}$$

and claim that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\}.$$

"$\subseteq$" Let $x \in Q$. Since Q is open, we have $p(x) < 1$. For every $f \in \Upsilon$ we have $f(x) \leq p(x)$ by definition. This proves $f(x) < 1$ for every $f \in \Upsilon$.

"$\supseteq$" Suppose $x_0 \notin Q$. We hope to find some $f \in \Upsilon$ with $f(x_0) \geq 1$. Towards that end, we define the functional

$$\begin{aligned} \ell\colon \operatorname{span}(\{x_0\}) &\to \mathbb{R} \\ t x_0 &\mapsto t. \end{aligned}$$

Since Q is convex and contains the origin, we have $p(x_0) \geq 1$. In particular, we have

$$\begin{aligned} \forall t \geq 0: &\quad \ell(tx_0) = t \leq t\, p(x_0) = p(tx_0), \\ \forall t < 0: &\quad \ell(tx_0) = t < 0 \leq p(tx_0). \end{aligned}$$

The Hahn-Banach theorem implies that there exists a linear functional $f\colon X \to \mathbb{R}$ which agrees with ℓ on $\operatorname{span}(\{x_0\})$ and satisfies $f(x) \leq p(x)$ for every $x \in X$. Is f continuous? Since Q is open and contains the origin, there exists $r > 0$ such that $B_r(0) \subset Q$. Thus, $\frac{1}{\lambda}x \in Q$ with $\lambda = \frac{2}{r}\|x\|_X$ and the definition of p implies that

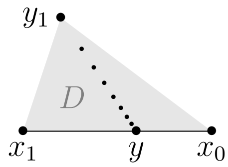
$$f(x) \leq p(x) \leq \tfrac{2}{r}\|x\|_X$$

which yields that f is continuous and therefore $f \in \Upsilon$. Since $f(x_0) = 1$, the claim follows.

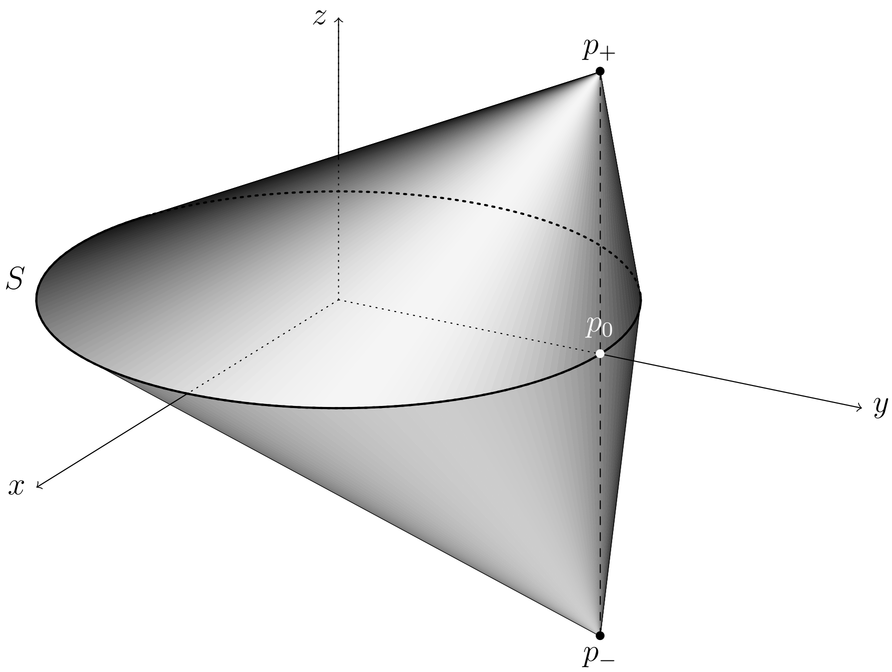
9.2. Extremal points

(a) It is clear that the set E of extremal points of the closed, convex subset $K \subset \mathbb{R}^2$ must be a subset of the boundary ∂K of K because the center of every ball contained in K is a convex combination of other points in this ball.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E which converges to some $y \in K$. Suppose $y \notin E$. Then there exist distinct points $x_1, x_0 \in K$ and some $0 < \lambda < 1$ such that $\lambda x_1 + (1-\lambda)x_0 = y$. For any $n \in \mathbb{N}$, the point y_n is extremal and therefore cannot lie on the segment between x_1 and x_0. Intuitively, the sequence $(y_n)_{n\in\mathbb{N}}$ must approach y from "above" or "below" this segment. By restriction to a subsequence, we can assume that all y_n strictly lie on the same side of the the affine line through x_1 and x_2. By convexity of K, the triangle $D = \operatorname{conv}\{x_1, x_0, y_1\}$ is a subset of K. The arguments above and convergence $y_n \to y$ imply that for $n \in \mathbb{N}$ sufficiently large, y_n is in the interior of D and thus in the interior of K. This however contradicts $y_n \in E \subset \partial K$. We conclude $y \in E$ which proves that E is closed.



(b) The set of extremal points of a closed, convex subset in $\mathbb{R}^3$ is not necessarily closed: Let $S = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $p_\pm = (0, 1, \pm 1)$. The set of extremal points of $\operatorname{conv}(S \cup \{p_+, p_-\})$ is $E = \{p_+, p_-\} \cup S \setminus p_0$, where $p_0 = (0, 1, 0) = \frac{1}{2}p_+ + \frac{1}{2}p_-$.



9.3. Extremal subsets

(a) Let $K \subset X$ be convex and $M \subset K$ an extremal subset of K. Suppose, $K \setminus M$ is not convex. Then there are points $x_1, x_0 \in K \setminus M$ such that $x := \lambda x_1 + (1-\lambda)x_0 \notin K \setminus M$ for some $0 < \lambda < 1$. Since K is convex, $x \in K$ and hence $x \in M$. However, this contradicts $x_1, x_0 \notin M$ by definition of extremal subset.

(b) No, the interval $K = [-1, 1] \subset \mathbb{R}$, the subset $N = [-1, 0] \subset K$ and the difference $K \setminus N =]0, 1]$ are all convex but N is not an extremal subset of K since $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0 \in N$ but $1 \notin N$.

(c) If $y \in K$ is an extremal point of K, then $\{y\} \subset K$ is an extremal subset of K and (a) implies that $K \setminus \{y\}$ is convex. Conversely, if $y \in K$ is not an extremal point of K, then by definition there exist $x_1, x_0 \in K \setminus \{y\}$ and some $0 < \lambda < 1$ such that $y = \lambda x_1 + (1-\lambda)x_0$ which shows that $K \setminus \{y\}$ is not convex.

9.4. Weak sequential continuity of linear operators

"(i) ⇒ (ii)" Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\mathrm{w}} x$ for some $x \in X$. Let $f \in Y^*$ be arbitrary. If $T\colon X \to Y$ is a continuous linear operator, then $f \circ T \in X^*$ and weak convergence of $(x_n)_{n\in\mathbb{N}}$ implies

$$\lim_{n\to\infty} f(Tx_n) = \lim_{n\to\infty} (f \circ T)(x_n) = (f \circ T)(x) = f(Tx)$$

which proves weak convergence of $(Tx_n)_{n\in\mathbb{N}}$ in Y.

"(ii) ⇒ (i)" If the linear operator $T\colon X \to Y$ is not continuous, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\|x_n\|_X \leq 1$ and $\|Tx_n\|_Y \geq n^2$ for every $n \in \mathbb{N}$. Then $\frac{1}{n}x_n \to 0$ in X (in particular weakly) but $(T(\frac{1}{n}x_n))_{n\in\mathbb{N}}$ is unbounded in Y and therefore cannot be weakly convergent (Satz 4.6.1.).

9.5. Weak convergence in finite dimensions

Let $e_1, \ldots, e_d$ be a basis for the finite-dimensional normed space $(X, \|\cdot\|_X)$. Then, every element $x \in X$ is of the form $x = \sum_{k=1}^{d} x^k e_k$ for uniquely determined $x^1, \ldots, x^d \in \mathbb{R}$ (upper indices, no exponents). For $k \in \{1, \ldots, d\}$ we consider the linear maps $e_k^*\colon X \to \mathbb{R}$ given by $e_k^*(x) = x^k$. In fact, $e_k^* \in X^*$ since $|e_k^*(x)| = |x^k| \leq \|x\|_1$, where $\|x\|_1 := \sum_{k=1}^{d} |x^k|$ defines a norm on X which must be equivalent to $\|\cdot\|_X$ since X is finite-dimensional.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \xrightarrow{\mathrm{w}} x$ for some $x \in X$ as $n \to \infty$, then

$$\forall k \in \{1, \ldots, d\} : \quad \lim_{n\to\infty} x_n^k = \lim_{n\to\infty} e_k^*(x_n) = e_k^*(x) = x^k.$$

This implies $\|x_n - x\|_1 \to 0$ and by equivalence of norms $\|x_n - x\|_X \to 0$ as $n \to \infty$.

9.6. Weak convergence in Hilbert spaces

(a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the Hilbert space $(H,(\cdot,\cdot)_H)$ such that $x_n \xrightarrow{w} x$ for some $x \in H$ and such that $\|x_n\|_H \to \|x\|_H$ as $n\to\infty$. Since $(x,\cdot)_H \in H^*$, weak convergence implies $(x,x_n)_H \to (x,x)_H = \|x\|_H^2$ as $n\to\infty$ and we have

$$\|x_n - x\|_H^2 = (x_n - x, x_n - x)_H = \|x_n\|_H^2 - 2(x,x_n)_H + \|x\|_H^2 \xrightarrow{n\to\infty} 0.$$

(b) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in H and $x,y\in H$ such that $x_n \xrightarrow{w} x$ and $\|y_n - y\|_H \to 0$ as $n\to\infty$. Weak convergence $x_n \xrightarrow{w} x$ implies in particular, that $(x_n,y)_H \to (x,y)_H$ as $n\to\infty$ and that there exists a finite constant C such that $\|x_n\|_H \le C$ for all $n\in\mathbb{N}$. Thus,

$$\begin{aligned}\Big|(x_n,y_n)_H - (x,y)_H\Big| &= \Big|(x_n,y_n-y)_H + (x_n,y)_H - (x,y)_H\Big| \\ &\le C\|y_n - y\|_H + \Big|(x_n,y)_H - (x,y)_H\Big| \xrightarrow{n\to\infty} 0.\end{aligned}$$

(c) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of the infinite-dimensional Hilbert space $(H,(\cdot,\cdot)_H)$. Then, Bessel's inequality

$$\sum_{n=0}^{\infty} |(x,e_n)_H|^2 \le \|x\|_H^2$$

implies $(x,e_n)_H \to 0$ as $n\to\infty$ for any $x\in H$. Since by the Riesz representation theorem any $f\in H^*$ satisfies $f(e_n) = (x,e_n)_H$ for a unique $x\in H$, we obtain $e_n \xrightarrow{w} 0$.

(d) Let $x\in H$ satisfy $\|x\|_H \le 1$. If $x = 0$, then any orthonormal system converges weakly to x by (c). If $x\ne 0$, then an orthonormal system $(e_n)_{n\in\mathbb{N}}$ of H with $e_1 = \|x\|_H^{-1}x$ can be constructed via the Gram-Schmidt algorithm. For $n\in\mathbb{N}$, let

$$x_n := x + \Big(\sqrt{1-\|x\|_H^2}\Big)e_{n+1}$$

Then, since $x \perp e_{n+1}$, we have $\|x_n\|^2 = \|x\|_H^2 + (1-\|x\|_H^2) = 1$ for every $n\in\mathbb{N}$. Moreover, $x_n \xrightarrow{w} x$ follows from $e_{n+1} \xrightarrow{w} 0$ as $n\to\infty$ by (c).

(e) Let $f_n\colon [0,2\pi]\to\mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n\in\mathbb{N}$. Then, $(\sqrt{\frac{1}{\pi}}f_n)_{n\in\mathbb{N}}$ is an orthonormal system of $L^2([0,2\pi])$, because

$$\begin{aligned}\int_0^{2\pi} \sin(mt)\sin(nt)\,\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi} \cos\big((m-n)t\big) - \cos\big((m+n)t\big)\,\mathrm{d}t \\ &= \begin{cases} 0, & \text{if } m\ne n, \\ \pi, & \text{if } m = n \end{cases}\end{aligned}$$

holds for any $m,n\in\mathbb{N}$. By (c) weak convergence $f_n \xrightarrow{w} 0$ as $n\to\infty$ follows.