## 10.1. Project: The weak topology is not metrizable

**(a)** Let $(X,\tau)$ be a metrizable topological space. Let $d\colon X\times X\to\mathbb{R}$ be a metric inducing the topology $\tau$. Given $x\in X$, we consider

$$B_\varepsilon(x) := \{y\in X \mid d(x,y)<\varepsilon\}, \qquad\qquad \mathcal{B}_x := \{B_{\frac{1}{n}}(x) \mid n\in\mathbb{N}\}.$$

Let $U$ be any neighbourhood of $x$. Since $(X,\tau)$ is metrizable, there exists $\varepsilon>0$ such that $B_\varepsilon(x)\subset U$. Choosing $\mathbb{N}\ni n>\frac{1}{\varepsilon}$, we have $B_{\frac{1}{n}}(x)\subset U$, which shows that $\mathcal{B}_x$ is a neighbourhood basis of $x$ in $(X,\tau)$. Since $x\in X$ is arbitrary and $\mathcal{B}_x$ countable, we have verified the first axiom of countability for $(X,\tau)$.

**(b)** Let $(X,\|\cdot\|_X)$ be a normed space. Let $\tau_{\mathrm{w}}$ be the weak topology on $X$. Let $U\subset X$ be any neighbourhood of $0\in X$ in $(X,\tau_{\mathrm{w}})$. Then there exists $\Omega\in\tau_{\mathrm{w}}$ such that $0\in\Omega\subset U$. By definition of weak topology, $\Omega$ is an arbitrary union and finite intersection of sets of the form $f^{-1}(I)$ for $f\in X^*$ and $I\subset\mathbb{R}$ open. In particular, $\Omega$ contains a finite intersection of such sets containing the origin. More precisely, there exist $f_1,\dots,f_n\in X^*$ and open sets $I_1,\dots,I_n\subset\mathbb{R}$ such that

$$\Omega\supset\bigcap_{k=1}^n f_k^{-1}(I_k)\ni 0.$$

By linearity $f_k(0)=0\in I_k$ for every $k\in\{1,\dots,n\}$. Since $I_1,\dots,I_n\subset\mathbb{R}$ are open and $n$ finite, there exists $\varepsilon>0$ such that $(-\varepsilon,\varepsilon)\subset I_k$ for every $k\in\{1,\dots,n\}$. Thus,

$$\Omega\supset\bigcap_{k=1}^n f_k^{-1}\big((-\varepsilon,\varepsilon)\big)=\{x\in X \mid \forall k=1,\dots,n:\ |f_k(x)|<\varepsilon\}$$

and we conclude that a neighbourhood basis of $0\in X$ in $(X,\tau_{\mathrm{w}})$ is given by

$$\mathcal{B}:=\bigg\{\bigcap_{k=1}^n f_k^{-1}\big((-\varepsilon,\varepsilon)\big)\ \bigg|\ n\in\mathbb{N},\ f_1,\dots,f_n\in X^*,\ \varepsilon>0\bigg\}.$$

**(c)** Let $f_1,\dots,f_n\in X^*$ and $f\in X^*$ be given. Suppose,

$$f(x)=0 \qquad \forall x\in N:=\{x\in X \mid f_1(x)=\ldots=f_n(x)=0\} \tag{$*$}$$

Let the linear map $\varphi\colon X\to\mathbb{R}^n$ be defined by

$$\varphi(x)=\big(f_1(x),\dots,f_n(x)\big).$$

Assumption $(*)$ implies $\ker\varphi\subset\ker f$. Let $F\colon X/\ker\varphi\cong\operatorname{im}(\varphi)\subset\mathbb{R}^n\to\mathbb{R}$ be defined by $F([x]):=f(x)$. This is well-defined since $F([x+p])=f(x)+f(p)=f(x)$ for every $p\in\ker\varphi\subset\ker f$. Defining $F$ to be zero on the orthogonal complement of

$\mathrm{im}(\varphi) \subset \mathbb{R}^n$, we obtain a linear map $F\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $f = F \circ \varphi$. By the Riesz representation theorem on $\mathbb{R}^n$ we have $F(y_1, \dots, y_n) = \lambda_1 y_1 + \ldots + \lambda_n y_n$ for some $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. This implies

$$f(x) = F(\varphi(x)) = \lambda_1 f_1(x) + \ldots + \lambda_n f_n(x).$$

Conversely, if $f$ is a linear combination of $\{f_1, \dots, f_n\}$, then $f(x) = 0$ for every $x \in N$.

**(d)** Let $(X, \|\cdot\|_X)$ be a normed space and suppose that $(X, \tau_{\mathrm{w}})$ is first countable. Then there exists a countable neighbourhood basis $\{A_\alpha\}_{\alpha\in\mathbb{N}}$ of $0 \in X$ in $(X, \tau_{\mathrm{w}})$. Since $\mathcal{B}$ defined in (b) is also a neighbourhood basis of $0 \in X$ in $(X, \tau_{\mathrm{w}})$, we have

$$\forall \alpha \in \mathbb{N} \quad \exists B_\alpha \in \mathcal{B}: \quad B_\alpha \subset A_\alpha.$$

By construction of $\mathcal{B}$, this means that

$$\forall \alpha \in \mathbb{N} \quad \exists n_\alpha \in \mathbb{N},\ f_1^\alpha, \dots, f_{n_\alpha}^\alpha \in X^*,\ \varepsilon_\alpha > 0 :$$
$$B_\alpha := \{x \in X \mid \forall k = 1, \dots, n_\alpha :\ |f_k^\alpha(x)| < \varepsilon_\alpha\} \subset A_\alpha.$$

We claim that every $f \in X^*$ is a finite linear combination of elements in the set

$$\Upsilon := \bigcup_{\alpha\in\mathbb{N}} \{f_k^\alpha \mid k = 1, \dots, n_\alpha\}.$$

Let $f \in X^*$. Then, $\{x \in X \mid |f(x)| < 1\}$ is a neighbourhood of $0 \in X$ in $(X, \tau_{\mathrm{w}})$. Consequently, there exists $\alpha \in \mathbb{N}$ such that $A_\alpha \subset \{x \in X \mid |f(x)| < 1\}$. Then, for every $m > 0$ by linearity

$$\{x \in X \mid \forall k = 1, \dots, n_\alpha :\ |f_k^\alpha(x)| < \tfrac{1}{m}\varepsilon_\alpha\}$$
$$= \tfrac{1}{m} B_\alpha \subset \tfrac{1}{m} A_\alpha \subset \{\tfrac{1}{m} x \in X \mid |f(x)| < 1\} = \{x \in X \mid |f(x)| < \tfrac{1}{m}\}.$$

Taking the intersection over all $m \in \mathbb{N}$, we obtain

$$\{x \in X \mid \forall k = 1, \dots, n_\alpha :\ f_k^\alpha(x) = 0\} \subset \{x \in X \mid f(x) = 0\}.$$

According to part (c), this implies that $f$ is a linear combination of $\{f_1^\alpha, \dots, f_{n_\alpha}^\alpha\}$ which is a finite subset of $\Upsilon$. Since $\Upsilon \subset X^*$ is at most countable, an algebraic basis for $X^*$ is at most countable.

**(e)** Suppose $(X, \tau_{\mathrm{w}})$ is metrizable. Then $(X, \tau_{\mathrm{w}})$ satisfies the first axiom of countability according to part (a). According to part (d), an algebraic basis for $X^*$ is at most countable. However, $(X^*, \|\cdot\|_{X^*})$ is always complete because $\mathbb{R}$ is complete (Beispiel 2.1.1). In problem 4.1 (a) we applied the Baire Lemma to show that an algebraic basis of a complete space is either finite or uncountable. If the algebraic basis of $X^*$ is finite, then $\infty > \dim X^* = \dim X^{**} \geq \dim X$ which contradicts our assumption. Therefore $(X, \tau_{\mathrm{w}})$ can not be metrizable.

### 10.2. Sequential closure

**(a)** Let $(X,\tau)$ be a topological space and let $A\subset X$ be closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $A$ such that $x_n \xrightarrow{\tau} x$ as $n\to\infty$ for some $x\in X$. Suppose $x\notin A$. Then, $U := X\setminus A$ is an open set in $\tau$ containing $x$. Convergence $x_n \xrightarrow{\tau} x$ implies that there exists $N\in\mathbb{N}$ such that $x_N\in U$. This however contradicts $x_N\in A$. Thus, $x\in A$ and we have proven that $A$ is sequentially closed.

Similarly, if $\Omega\subset X$ is any subset and $x\in\overline{\Omega}_{\tau\text{-seq}}$, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\Omega$ such that $x_n \xrightarrow{\tau} x$. If $x\notin\overline{\Omega}_\tau$, then $U := X\setminus\overline{\Omega}_\tau$ is an open set in $\tau$ containing $x$. Convergence $x_n\xrightarrow{\tau} x$ implies that there exists $N\in\mathbb{N}$ such that $x_N\in U$ in contradiction to $x_N\in\Omega\subset\overline{\Omega}_\tau$. Thus, $x\in\overline{\Omega}_\tau$ and the inclusion $\overline{\Omega}_{\tau\text{-seq}}\subset\overline{\Omega}_\tau$ follows.

**(b)** In the following, we construct a set $\Omega\subset\ell^2$ such that $(0)\in\overline{\Omega}_{\mathrm{w}}$ but no sequence in $\Omega$ converges weakly to zero: $(0)\notin\overline{\Omega}_{\text{w-seq}}$. (Here we denote $(0) := (0,0,\ldots)\in\ell^2$.)

For $n\in\mathbb{N}$ and $2\le m\in\mathbb{N}$, let $x^{(n,m)} = (\frac{1}{n},0,\ldots,0,n,0,\ldots)\in\ell^2$, where the entry "$n$" is at $m$-th position. By the Riesz representation theorem, any $f\in(\ell^2)^*$ is of the form $f=(\cdot,y)_{\ell^2}$ for some $y\in\ell^2$. For any $y\in\ell^2$ and any $2\le m,n\in\mathbb{N}$, we have

$$(x^{(n,m)},y)_{\ell^2} = \tfrac{1}{n}y_1 + ny_m. \tag{$*$}$$

Let $\Omega=\{x^{(n,m)}\mid n,m\in\mathbb{N},\ m\ge 2\}$. Let $(x^{(n_k,m_k)})_{k\in\mathbb{N}}$ be any (fixed) sequence in $\Omega$. Towards a contradiction, suppose $x^{(n_k,m_k)}\xrightarrow{\mathrm{w}}(0)$ as $k\to\infty$. From $(*)$ we conclude $n_k\to\infty$ and $m_k\to\infty$ as $k\to\infty$. (Note that for $y\in\ell^2$ we have $y_m\to 0$ as $m\to\infty$.) But then $\|x^{(n_k,m_k)}\|_{\ell^2}^2 = n_k^{-2}+n_k^2\to\infty$ as $k\to\infty$ and we derived a contradiction to the fact, that $(x^{(n_k,m_k)})_{k\in\mathbb{N}}$ being a weakly convergent sequence must be bounded.

Suppose, $(0)\notin\overline{\Omega}_{\mathrm{w}}$. Then there exists a weak neighbourhood $V$ of $(0)\in\ell^2$ such that $V\subset\ell^2\setminus\overline{\Omega}_{\mathrm{w}}$. By definition of weak topology, there exist finitely many open sets $U_1,\ldots,U_r\subset\mathbb{R}$ and elements $y^{(1)},\ldots,y^{(r)}\in\ell^2$, where $y^{(k)}=(y_j^{(k)})_{j\in\mathbb{N}}$ such that

$$V\supset\bigcap_{k=1}^r\{x\in\ell^2\mid (x,y^{(k)})_{\ell^2}\in U_k\}\ni(0).$$

In particular we have $0\in U_k$ for every $k\in\{1,\ldots,r\}$. Since every $U_k$ is open and $r$ finite, there exists $\varepsilon>0$ such that $(-\varepsilon,\varepsilon)\subset U_k$ for every $k\in\{1,\ldots,r\}$. However, if we fix $n\in\mathbb{N}$ such that $\frac{1}{n}|y_1^{(k)}|<\frac{\varepsilon}{2}$ and then choose $2\le m\in\mathbb{N}$ large enough such that $n|y_m^{(k)}|<\frac{\varepsilon}{2}$ for each of the finitely many $k\in\{1,\ldots,r\}$, we have

$$|(x^{(n,m)},y^{(k)})_{\ell^2}|\le\tfrac{1}{n}|y_1^{(k)}|+n|y_m^{(k)}|<\varepsilon\qquad\forall k\in\{1,\ldots,r\}$$

which implies $x^{(n,m)}\in V$. As $x^{(n,m)}\in\Omega$, a contradiction to the definition of $V$ arises.

### 10.3. Convex hull

**(a)** Given the normed space $(X, \|\cdot\|_X)$ and the subset $A \subset X$, let

$$\mathcal{C} := \left\{ \sum_{k=1}^{n} \lambda_k x_k \;\middle|\; n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1 \right\}.$$

We prove $\operatorname{conv}(A) = \mathcal{C}$ by showing the two inclusions.

"$\subseteq$" Since $A \subset \mathcal{C}$, the inclusion $\operatorname{conv}(A) \subseteq \mathcal{C}$ follows from the definition of convex hull, if we show that $\mathcal{C}$ is convex. In fact, given $0 < t < 1$ we have

$$t \sum_{k=1}^{n} \lambda_k x_k + (1-t) \sum_{k=1}^{m} \lambda'_k x'_k = \sum_{k=1}^{n+m} \mu_k y_k$$

with

$$0 \leq \mu_k := \begin{cases} t\lambda_k & \text{if } k \in \{1, \ldots, n\}, \\ (1-t)\lambda'_{k-n} & \text{if } k \in \{n+1, \ldots, n+m\} \end{cases}$$

$$A \ni y_k := \begin{cases} x_k & \text{if } k \in \{1, \ldots, n\}, \\ x'_{k-n} & \text{if } k \in \{n+1, \ldots, n+m\} \end{cases}$$

and $\mu_1 + \ldots + \mu_{n+m} = t(\lambda_1 + \ldots + \lambda_n) + (1-t)(\lambda'_1 + \ldots + \lambda'_m) = t + (1-t) = 1$.

"$\supseteq$" Let $x_1, \ldots, x_n \in A$ and let $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \ldots + \lambda_n = 1$. We can assume $\lambda_1 \neq 0$. Since $\operatorname{conv}(A)$ is convex and contains $x_1, x_2 \in A$, and since $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$,

$$\operatorname{conv}(A) \ni \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} =: y_2.$$

For the same reason,

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} y_2 + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} x_3 = \frac{\lambda_1 x_2 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1 + \lambda_2 + \lambda_3} =: y_3.$$

Iterating this procedure, we obtain

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \ldots + \lambda_{k-1}}{\lambda_1 + \ldots + \lambda_k} y_{k-1} + \frac{\lambda_k}{\lambda_1 + \ldots + \lambda_k} x_k = \frac{\lambda_1 x_1 + \ldots + \lambda_k x_k}{\lambda_1 + \ldots + \lambda_k} =: y_k.$$

for every $k \in \{3, \ldots, n\}$. Since $\lambda_1 + \ldots + \lambda_n = 1$, we have $y_n = \lambda_1 x_1 + \ldots + \lambda_n x_n$ which concludes the proof of $\operatorname{conv}(A) \supseteq \mathcal{C}$.

**(b)** Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $X$ and let $x\in X$ such that $x_k \xrightarrow{\mathrm{w}} x$ as $k\to\infty$. Let $K:=\operatorname{conv}(\{x_k \mid k\in\mathbb{N}\})$. In general, $K\subset\overline{K}\subset\overline{K}_{\text{w-seq}}\subset\overline{K}_{\mathrm{w}}$ but since $K$ is convex, the closure $\overline{K}$ with respect to $\|\cdot\|_X$ agrees with the closure $\overline{K}_{\mathrm{w}}$ with respect to the weak topology: $\overline{K}=\overline{K}_{\mathrm{w}}$. Therefore, the assumption that $x$ is in the weak-sequential closure $\overline{K}_{\text{w-seq}}\ni x$ implies $x\in\overline{K}$ and there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $K$ such that $\|y_n-x\|_X\to 0$ as $n\to\infty$. By (a), each element $y_n\in K$ must be a convex linear combination of finitely many elements of $\{x_k \mid k\in\mathbb{N}\}$.

**(c)** Given the normed space $(X,\|\cdot\|_X)$, the convex subsets $A,B\subset X$ and defining $\triangle:=\{(s,t)\in\mathbb{R}^2 \mid s+t=1,\ s,t\geq 0\}$, we claim that

$$\operatorname{conv}(A\cup B)=\mathcal{D}:=\bigcup_{(s,t)\in\triangle}(sA+tB)$$

"$\subseteq$" By choosing $(s,t)=(1,0)$ we see $A\subset\mathcal{D}$. Analogously, $B\subset\mathcal{D}$, hence $A\cup B\subset\mathcal{D}$. If $x\in(\operatorname{conv}(A\cup B))\setminus(A\cup B)$, then (a) implies that $x$ is of the form

$$x=\sum_{k=1}^{j}s_k a_k+\sum_{k=j+1}^{n}t_k b_k,$$

where $j,n\in\mathbb{N}$, where $a_k\in A$ and $b_k\in B$ as well as $s_k,t_k\geq 0$ for every $k$ and where $s_1+\ldots+s_j+t_{j+1}+\ldots+t_n=1$. Since $x\notin A\cup B$ by assumption, we have

$$s:=\sum_{k=1}^{j}s_k>0, \qquad\qquad t:=\sum_{k=j+1}^{n}t_k>0,$$

with $s+t=1$. Since $A$ and $B$ are both convex by assumption,

$$a:=\frac{1}{s}\sum_{k=1}^{j}s_k a_k\in A, \qquad\qquad b:=\frac{1}{t}\sum_{k=j+1}^{n}t_k b_k\in B,$$

and we have shown $x=sa+tb\in\mathcal{D}$.

"$\supseteq$" Let $a\in A$ and $b\in B$. Then $a,b\in\operatorname{conv}(A\cup B)$. Since $\operatorname{conv}(A\cup B)$ is convex, we must have $sa+tb\in\operatorname{conv}(A\cup B)$ for every $(s,t)\in\triangle$. This proves $\operatorname{conv}(A\cup B)\supseteq\mathcal{D}$.

Under the assumption that the convex sets $A$ and $B$ are compact, we show now that

$$\mathcal{D}=\bigcup_{(s,t)\in\triangle}(sA+tB)$$

is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$. Then there exist $a_n\in A$ and $b_n\in B$ as well as $(s_n,t_n)\in\triangle$ such that $x_n=s_na_n+t_nb_n$ for every $n\in\mathbb{N}$. We argue in 3 steps:

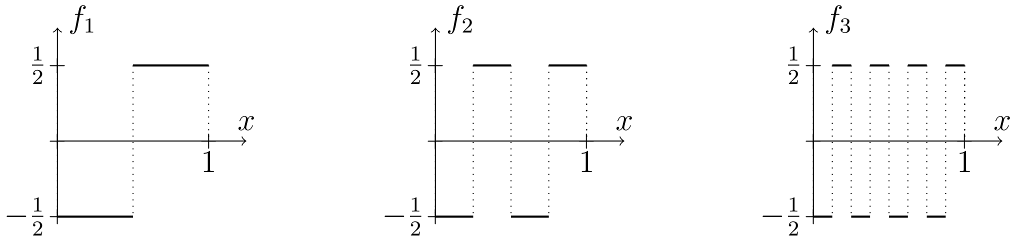
- Since $\triangle$ is compact in $\mathbb{R}^2$, a subsequence $((s_n,t_n))_{n\in\Lambda_1\subset\mathbb{N}}$ converges in $\triangle$.
- Since $A$ is compact in $X$, a subsequence $(a_n)_{n\in\Lambda_2\subset\Lambda_1}$ converges in $A$.
- Since $B$ is compact in $X$, a subsequence $(b_n)_{n\in\Lambda_3\subset\Lambda_2}$ converges in $B$.

Therefore, the subsequence $(x_n)_{n\in\Lambda_3}$ converges in $\mathcal{D}$ which concludes the proof.

### 10.4. Non-compactness

**(a)** Given $n \in \mathbb{N}$, we divide the interval $[0,1]$ into $2^n$ subintervals $I_1, \ldots, I_{2^n}$ of equal length, and define the function $f_n \colon [0,1] \to \mathbb{R}$ on each $I_k$ to be $-\frac{1}{2}$ if $k$ is odd and $+\frac{1}{2}$ if $k$ is even. More precisely,

$$f_n(x) = \begin{cases} -\frac{1}{2}, & \text{if } \exists k \in \mathbb{N}: \ 2^n x \in [2k-2, 2k-1[ \\ \frac{1}{2}, & \text{else.} \end{cases}$$




By construction, $\|f_n\|_{L^p([0,1])} = \frac{1}{2}$ for every $n \in \mathbb{N}$ and every $1 \leq p \leq \infty$. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p([0,1])$. However by construction, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ the difference $|f_n - f_m|$ is the characteristic function of a union of subintervals whose lengths sum up to $\frac{1}{2}$. In particular, $\|f_n - f_m\|_{L^p([0,1])} = (\frac{1}{2})^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f_n - f_m\|_{L^\infty([0,1])} = 1$. Consequently, $(f_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

**(b)** Given $n \in \mathbb{N}$, let $e_n \in c_0$ be given by $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is at $n$-th position. Then the sequence $(e_n)_{n\in\mathbb{N}}$ is bounded in $(c_0, \|\cdot\|_{\ell^\infty})$ since $\|e_n\|_{\ell^\infty} = 1$ for every $n \in \mathbb{N}$. However, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ we have $\|e_n - e_m\|_{\ell^\infty} = 1$. Consequently, $(e_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

### 10.5. Separability

The claim is equivalence of the following statements.

(i) The normed space $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.

Since subsets of separable sets are separable (Satz 5.2.1), the inclusions $S \subset B \subset X$ already yield (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

"(iii) $\Rightarrow$ (i)" By assumption, there exists a countable dense subset $D \subset S$. Moreover, as countable union of countable sets,

$$E := \bigcup_{q\in\mathbb{Q}} qD = \{qd \in X \mid q \in \mathbb{Q},\ d \in D\}$$

is countable. We claim is that $E \subset X$ is dense. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Since $0 \in E$, we may assume $x \neq 0$ and consider the element $\frac{x}{\|x\|_X} \in S$. Since $D \subset S$ is dense, there exists $d \in D$ such that

$$\Big\|d - \tfrac{x}{\|x\|_X}\Big\|_X < \frac{\varepsilon}{2\|x\|_X}.$$

Moreover, since $\|x\|_X \in \mathbb{R}$ and since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that

$$\Big|q - \|x\|_X\Big| < \frac{\varepsilon}{2}.$$

Using $D \subset S \Rightarrow \|d\|_X = 1$ and combining the inequalities, the point $qd \in E$ satisfies

$$\begin{aligned}\|qd - x\|_X &= \Big\|(q - \|x\|_X)d + \|x\|_X d - x\Big\|_X \\ &\leq \Big|q - \|x\|_X\Big| + \Big\|\|x\|_X d - x\Big\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon\|x\|_X}{2\|x\|_X} = \varepsilon,\end{aligned}$$

which proves that $E \subset X$ is dense. Since $E$ is countable, we have shown that $X$ is separable.