11.1. Dual Operators

(a) Let $x \in X$ and $x^* \in X^*$ be arbitrary. By definition of $(\mathrm{Id}_X)^* \colon X^* \to X^*$, we have

$$\Big\langle (\mathrm{Id}_X)^* x^*, x \Big\rangle_{X^* \times X} = \Big\langle x^*, \mathrm{Id}_X\, x \Big\rangle_{X^* \times X} = \langle x^*, x \rangle_{X^* \times X^*}.$$

Since $x \in X$ is arbitrary, $(\mathrm{Id}_X)^* x^* = x^*$. Since $x^* \in X^*$ is arbitrary, $(\mathrm{Id}_X)^* = \mathrm{Id}_{(X^*)}$.

(b) Let $z^* \in Z^*$ and $x \in X$ be arbitrary. Then, $(S \circ T)^* = T^* \circ S^*$ follows from

$$\begin{aligned}\Big\langle (S \circ T)^* z^*, x \Big\rangle_{X^* \times X} &= \Big\langle z^*, S(Tx) \Big\rangle_{Z^* \times Z} \\ &= \Big\langle S^* z^*, Tx \Big\rangle_{Y^* \times Y} = \Big\langle T^*(S^* z^*), x \Big\rangle_{X^* \times X}.\end{aligned}$$

(c) To prove $(T^*)^{-1} = (T^{-1})^*$, we apply the results from (a) and (b) and obtain

$$\begin{aligned}&T^* \circ (T^{-1})^* = (T^{-1} \circ T)^* = (\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}, \\ &(T^{-1})^* \circ T^* = (T \circ T^{-1})^* = (\mathrm{Id}_Y)^* = \mathrm{Id}_{Y^*}\,.\end{aligned}$$

(d) Let $x \in X$ and $y^* \in Y^*$ be arbitrary. Then, $(\mathcal{I}_Y \circ T) = (T^*)^* \circ \mathcal{I}_X$ follows from

$$\begin{aligned}\Big\langle (\mathcal{I}_Y \circ T)x, y^* \Big\rangle_{Y^{**} \times Y^*} &= \Big\langle Tx, y^* \Big\rangle_{Y \times Y^*} = \Big\langle x, T^* y^* \Big\rangle_{X \times X^*} \\ &= \Big\langle \mathcal{I}_X x, T^* y^* \Big\rangle_{X^{**} \times X^*} = \Big\langle (T^*)^* (\mathcal{I}_X x), y^* \Big\rangle_{Y^{**} \times Y^*}.\end{aligned}$$

11.2. Isomorphisms and Isometries

(a) The dual operator T^* of any $T \in L(X, Y)$ with $T^{-1} \in L(Y, X)$ is invertible according to problem 11.1 (c) and its inverse is $(T^*)^{-1} = (T^{-1})^*$. Moreover, the assumption $T^{-1} \in L(Y, X)$ implies $(T^{-1})^* \in L(X^*, Y^*)$. Hence, T^* is an isomorphism.

(b) If T is an isometric isomorphism, then T^* is an isomorphism by (a) and

$$\|T^* y^*\|_{X^*} = \sup_{\|x\|_X \le 1} \Big| \langle T^* y^*, x \rangle_{X^* \times X} \Big| = \sup_{\|Tx\|_Y = \|x\|_X \le 1} \Big| \langle y^*, Tx \rangle_{Y^* \times Y} \Big| = \|y^*\|_{Y^*}.$$

(c) If X and Y are reflexive, $\mathcal{I}_X \colon X \to X^{**}$ and $\mathcal{I}_Y \colon Y \to Y^{**}$ are bijective isometries. If T^* is an (isometric) isomorphism, then (a) and (b) imply that $(T^*)^*$ is an (isometric) isomorphism. Applying the result of problem 11.1 (d), we see that the same holds for

$$T = \mathcal{I}_Y^{-1} \circ (T^*)^* \circ \mathcal{I}_X.$$

(d) Since X is reflexive by assumption, $\mathcal{I}_X$ is an isomorphism. Suppose, $T \colon X \to Y$ is an isomorphism. Applying part (a) twice, $(T^*)^*$ is an isomorphism. Moreover,

$$\mathcal{I}_Y = (T^*)^* \circ \mathcal{I}_X \circ T^{-1}$$

according to 11.1 (d). Since $\mathcal{I}_Y$ is a composition of isomorphisms, Y is reflexive.

11.3. Minimal Energy

(a) Given a bounded domain $\Omega \subset \mathbb{R}^m$ and $g \in L^2(\mathbb{R}^m)$, the goal is weak sequential continuity of the map

$$\begin{aligned} V\colon L^2(\Omega) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y)\, dy\, dx \end{aligned}$$

Claim 1. The linear operator $T\colon L^2(\Omega) \to L^2(\Omega)$ mapping $f \mapsto Tf$ given by

$$(Tf)(x) = \int_\Omega g(x-y) f(y)\, dy$$

is well defined.

Proof. Let $f \in L^2(\Omega)$. Then, by Hölder's inequality

$$\begin{aligned} \|Tf\|^2_{L^2(\Omega)} &= \int_\Omega |(Tf)(x)|^2\, dx = \int_\Omega \Big| \int_\Omega g(x-y) f(y)\, dy \Big|^2 dx \\ &\leq \int_\Omega \Big(\int_\Omega |g(x-y) f(y)|\, dy \Big)^2 dx \leq \int_\Omega \Big(\int_\Omega |g(x-y)|^2\, dy \Big) \|f\|^2_{L^2(\Omega)}\, dx \\ &\leq \int_\Omega \|g\|^2_{L^2(\mathbb{R}^m)} \|f\|^2_{L^2(\Omega)}\, dx \leq |\Omega| \|g\|^2_{L^2(\mathbb{R}^m)} \|f\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $\Omega \subset \mathbb{R}^m$ being bounded has finite volume $|\Omega| < \infty$, the claim follows. $\square$

Claim 2. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ such that $f_k \xrightarrow{w} f$ in $L^2(\Omega)$ as $k \to \infty$. Then, $\|Tf_k - Tf\|_{L^2(\Omega)} \to 0$ as $k \to \infty$, where T is as in claim 1.

Remark. In view of problem 11.4 (e), claim 2 states that T is a *compact operator*.

Proof. Since the sequence $(f_k)_{k\in\mathbb{N}}$ is weakly convergent, it is bounded (by Satz 4.6.1): $\exists C > 0\ \forall k \in \mathbb{N}:\ \|f_k\|_{L^2(\Omega)} \leq C$. For every fixed $x_0 \in \Omega$ and $k \in \mathbb{N}$, there holds

$$\begin{aligned} |(Tf_k)(x_0)| &\leq \int_\Omega |g(x_0 - y) f_k(y)|\, dy \leq \Big(\int_\Omega |g(x_0-y)|^2\, dy \Big)^{\frac{1}{2}} \Big(\int_\Omega |f_k(y)|^2\, dy \Big)^{\frac{1}{2}} \\ &\leq \|g\|_{L^2(\mathbb{R}^m)} \|f_k\|_{L^2(\Omega)}. \end{aligned}$$

In particular, the map $f_k \mapsto (Tf_k)(x_0)$ is a linear, continuous functional $L^2(\Omega) \to \mathbb{R}$. Therefore, weak convergence $f_k \xrightarrow{w} f$ implies $(Tf_k)(x_0) \to (Tf)(x_0)$ as $k \to \infty$. In other words, the function Tf converges pointwise to Tf_k Moreover,

$$|(Tf_k)(x_0)| \leq \|g\|_{L^2(\mathbb{R}^m)} \|f_k\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\mathbb{R}^m)}$$

uniformly for every $k \in \mathbb{N}$. Since Ω is bounded, the constant $C\|g\|_{L^2(\mathbb{R}^m)}$ on the right right hand side is trivially in $L^2(\Omega)$. Hence, the claim follows by dominated convergence. $\square$

Claim 3. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ such that $f_k \overset{\mathrm{w}}{\rightharpoonup} f$ in $L^2(\Omega)$ as $k\to\infty$. Then, $V(f_k)\to V(f)$ as $k\to\infty$, i. e. V is weakly sequentially continuous.

Proof. Let T be as in claim 1. Since $f_k \overset{\mathrm{w}}{\rightharpoonup} f$ and $\|Tf_k \to Tf\|_{L^2(\Omega)}\to 0$ as $k\to\infty$ by claim 2, we conclude

$$V(f_k)=\int_\Omega f_k(x)\int_\Omega g(x-y)f_k(y)\,dy\,dx=\langle f_k, Tf_k\rangle_{L^2(\Omega)}\xrightarrow{k\to\infty}\langle f, Tf\rangle = V(f),$$

using the continuity property of scalar products proven in problem 9.6 (b). □

(b) Given V as in (a) with $0\le g\in L^2(\mathbb{R}^m)$ and $h\in L^2(\Omega)$ the claim is that the map

$$\begin{aligned} E\colon L^2(\Omega)&\to\mathbb{R}\\ f&\mapsto \|f-h\|^2_{L^2(\Omega)}+V(f)\end{aligned}$$

restricted to $L^2_+(\Omega)=\{f\in L^2(\Omega)\mid f(x)\ge 0 \text{ for almost every } x\in\Omega\}$ attains a global minimum. Since $L^2(\Omega)$ is reflexive (being a Hilbert space), we may invoke the direct method in the calculus of variations if we prove the following claims.

Claim 4. $L^2_+(\Omega)$ is non-empty and weakly sequentially closed.

Proof. Clearly, $L^2_+(\Omega)\ni 0$ is non-empty. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^2_+(\Omega)$ such that $f_k \overset{\mathrm{w}}{\rightharpoonup} f$ for some $f\in L^2(\Omega)$ as $k\to\infty$. Suppose $f\notin L^2_+(\Omega)$. Then there exists $U\subset\Omega$ with positive measure such that $f|_U<0$. In particular, we can test the weak convergence with the characteristic function χ_U to obtain the contradiction

$$0>\langle f,\chi_U\rangle_{L^2(\Omega)}=\lim_{k\to\infty}\langle f_k,\chi_U\rangle\ge 0. \qquad \square$$

Claim 5. $E\colon L^2_+(\Omega)\to\mathbb{R}$ is coercive and weakly sequentially lower semi-continuous.

Proof. Since $V(f)\ge 0$ if both $g\ge 0$ and $f\ge 0$ almost everywhere, we have

$$\begin{aligned} E(f)\ge\|f-h\|^2_{L^2(\Omega)}&\ge\|f\|^2_{L^2(\Omega)}-2\|f\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)}+\|h\|^2_{L^2(\Omega)}\\ &\ge\frac{1}{2}\|f\|^2_{L^2(\Omega)}-\|h\|^2_{L^2(\Omega)}\end{aligned}$$

for every $f\in L^2_+(\Omega)$ using Young's inequality $2ab\le\frac{1}{2}a^2+2b^2$. Since $h\in L^2(\Omega)$ is fixed, E is coercive.

By part (a), $f\mapsto V(f)$ is weakly sequentially lower semi-continuous. Moreover, every term on the right hand side of

$$\|f-h\|^2_{L^2(\Omega)}=\|f\|^2_{L^2(\Omega)}-2\langle f,h\rangle_{L^2(\Omega)}+\|h\|^2_{L^2(\Omega)}$$

is weakly sequentially lower semi-continuous in f since h is fixed. This proves the claim. □

11.4. Compact Operators

(a) The claim is that $T \in L(X,Y)$ is a compact operator, i. e. $\overline{T(B_1(0))} \subset Y$ is compact, if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

"$\Rightarrow$" Let $T \in L(X,Y)$ be a compact operator. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists $M > 0$ such that $\|x_n\|_X < M$ for all $n \in \mathbb{N}$. In particular, $\frac{1}{M}x_n \in B_1(0) \subset X$ and $\frac{1}{M}Tx_n \in T(B_1(0))$ for every $n \in \mathbb{N}$. Since $\overline{T(B_1(0))} \subset Y$ is compact, a subsequence $(\frac{1}{M}Tx_{n_k})_{k\in\mathbb{N}}$ converges in Y. Hence, $(Tx_{n_k})_{k\in\mathbb{N}}$ also converges.

"$\Leftarrow$" Conversely, let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{T(B_1(0))}$. For every $n \in \mathbb{N}$ there exists $y'_n \in T(B_1(0))$ such that $\|y_n - y'_n\|_Y \leq \frac{1}{n}$. Since there exists a sequence $(x'_n)_{n\in\mathbb{N}}$ in $B_1(0) \subset X$ such that $Tx'_n = y'_n$, a subsequence $y'_{n_k} \to y$ converges in Y as $k \to \infty$ by assumption. Since $\|y_{n_k} - y\|_Y \leq \|y_{n_k} - y'_{n_k}\| + \|y'_{n_k} - y\|_Y \to 0$ as $k \to \infty$ we conclude that a subsequence of $(y_n)_{n\in\mathbb{N}}$ converges. Being closed, $\overline{T(B_1(0))}$ must contain the limit y which proves that $\overline{T(B_1(0))}$ is compact, i. e. T is a compact operator.

(b) Part (a) and linearity of the limit imply that the set of compact operators $K(X,Y) \subset L(X,Y)$ is a linear subspace. To prove that this subspace is closed, let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $K(X,Y)$ such that $\|T_k - T\|_{L(X,Y)} \to 0$ for some $T \in L(X,Y)$ as $k \to \infty$. To show $T \in K(X,Y)$, consider a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X and choose the nested, unbounded subsets $\mathbb{N} \supset \Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$ such that $(T_k x_n)_{n\in\Lambda_k}$ is convergent in Y with limit point $y_k \in Y$. This is possible by (a), since T_k is a compact operator for every $k \in \mathbb{N}$. Let $\Lambda \subset \mathbb{N}$ be the corresponding diagonal sequence (i. e. the k-th number in Λ is the k-th number in Λ_k). By continuity of $\|\cdot\|_Y$, we can estimate

$$\|y_k - y_m\|_Y = \lim_{\Lambda \ni n \to \infty} \|T_k x_n - T_m x_n\|_Y \leq \|T_k - T_m\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X$$

for any $k, m \in \mathbb{N}$. Since $(T_k)_{k\in\mathbb{N}}$ is convergent in $L(X,Y)$, we conclude that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in Y. Since $(Y, \|\cdot\|_Y)$ is assumed to be complete, $y_k \to y$ for some $y \in Y$ as $k \to \infty$. We claim that $(Tx_n)_{n\in\Lambda}$ also converges to y which would finish the proof of $T \in K(X,Y)$ by (a). Let $\varepsilon > 0$. Choose a fixed $\kappa \in \mathbb{N}$ such that

$$\|T - T_\kappa\|_{L(X,Y)} < \frac{\varepsilon}{3 \sup\limits_{n\in\Lambda} \|x_n\|_X}, \qquad \|y_\kappa - y\|_Y \leq \frac{\varepsilon}{3}.$$

Since $T_\kappa x_n \to y_\kappa$ as $\Lambda \ni n \to \infty$, there exists $N \in \Lambda$ such that for every $\Lambda \ni n \geq N$ $\|T_\kappa x_n - y_\kappa\| \leq \frac{\varepsilon}{3}$. Finally, the claim follows from the estimate

$$\begin{aligned}\|Tx_n - y\|_Y &\leq \|Tx_n - T_\kappa x_n\|_Y + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y \\ &\leq \|T - T_\kappa\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y < \varepsilon\end{aligned}$$

which holds for every $\Lambda \ni n \geq N$.

(c) The image of $B_1(0)$ under $T \in L(X,Y)$ is bounded. If $T(X) \subset Y$ is of finite dimension, then $\overline{T(B_1(0))}$ is compact as a bounded, closed subset of $T(X)$.

(d) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in X.

Suppose T is a compact operator. Then, a subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y by (a). Since S is continuous, $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in Z, which by (a) proves that $S \circ T$ is a compact operator.

Suppose S is a compact operator. Since T is continuous, the sequence $(Tx_n)_{n\in\mathbb{N}}$ is bounded in Y. Then, a subsequence $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in Z by (a), which again proves that $S \circ T$ is a compact operator.

(e) Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in X. Since X is reflexive, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly in X by the Eberlein–Šmulian theorem. Then, $(Tx_{n_k})_{k\in\mathbb{N}}$ is norm-convergent in Y by assumption and (a) implies that T is a compact operator.