12.1. Integral operators

(a) Let $f \in L^2(\Omega)$. Then Hölder's inequality and Fubini's theorem imply

$$\begin{aligned}\int_\Omega |(Kf)(x)|^2\,dx &= \int_\Omega \Bigl|\int_\Omega k(x,y)f(y)\,dy\Bigr|^2 dx \leq \int_\Omega \Bigl(\int_\Omega |k(x,y)f(y)|\,dy\Bigr)^2 dx \\ &\leq \int_\Omega \Bigl(\int_\Omega |k(x,y)|^2\,dy\Bigr)\|f\|^2_{L^2(\Omega)}\,dx = \|k\|^2_{L^2(\Omega\times\Omega)}\|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $k \in L^2(\Omega\times\Omega)$ by assumption, $\|Kf\|_{L^2(\Omega)} \leq \|k\|_{L^2(\Omega\times\Omega)}\|f\|_{L^2(\Omega)} < \infty$ follows.

(b) Since the space $L^2(\Omega)$ is reflexive (which follows from being a Hilbert space), problem 11.4 (e) implies that $K\colon L^2(\Omega)\to L^2(\Omega)$ is a compact operator, if K maps weakly convergent sequences to norm-convergent sequences.

Let $(f_n)_{n\in\mathbb{N}}$ be sequence in $L^2(\Omega)$ such that $f_n \xrightarrow{\text{w}} f$ as $n\to\infty$ for some $f \in L^2(\Omega)$. Since $k \in L^2(\Omega\times\Omega)$, Fubini's theorem implies that $k(x,\cdot)\in L^2(\Omega)$ for almost every $x\in\Omega$. Weak convergence therefore implies

$$(Kf_n)(x) = \bigl\langle k(x,\cdot), f_n\bigr\rangle_{L^2(\Omega)} \xrightarrow{n\to\infty} \bigl\langle k(x,\cdot), f\bigr\rangle_{L^2(\Omega)} = (Kf)(x)$$

for almost every $x\in\Omega$. As weakly convergent sequence, $(f_n)_{n\in\mathbb{N}}$ is bounded: there exists $C\in\mathbb{R}$ such that $\|f_n\|_{L^2(\Omega)} \leq C$ for every $n\in\mathbb{N}$. By Hölder's inequality,

$$|(Kf_n)(x)| \leq \int_\Omega |k(x,y)f_n(y)|\,dy \leq \|k(x,\cdot)\|_{L^2(\Omega)}\|f_n\|_{L^2(\Omega)} \leq C\|k(x,\cdot)\|_{L^2(\Omega)}.$$

The assumption $k \in L^2(\Omega\times\Omega)$ and Fubini's theorem imply that the function $x\mapsto C\|k(x,\cdot)\|_{L^2(\Omega)}$ is in $L^2(\Omega)$. Thus, $(Kf_n)(x)$ is dominated by a function in $L^2(\Omega)$. Since $(Kf_n)(x)$ converges pointwise for almost every $x\in\Omega$ to a function in $L^2(\Omega)$, the dominated convergence theorem implies L^2-convergence $\|Kf_n - Kf\|_{L^2(\Omega)} \to 0$.

12.2. Uniform subconvergence

For every $n\in\mathbb{N}$ and $x\in[0,1]$, the assumptions $f_n'(0) = f_n(0)$ and $|f_n'(t)|\leq C$ imply

$$|f_n(x)| \leq |f_n(0)| + \int_0^x |f_n'(t)|\,dt = |f_n'(0)| + \int_0^x |f_n'(t)|\,dt \leq C + x\,C \leq 2C.$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^0([0,1])$. It is also equicontinuous:

$$|f_n(x) - f_n(y)| = \Bigl|\int_y^x f_n'(t)\,dt\Bigr| \leq C|x-y|,$$

hence $|f_n(x)-f_n(y)| < \varepsilon$ whenever $|x-y| < \delta := \frac{\varepsilon}{2C}$. By the Arzelà–Ascoli theorem, $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

12.3. Multiplication operators on complex-valued sequences

(a) Given $a \in \ell^\infty_{\mathbb{C}}$, let $(Tx)_n = a_n x_n$ for $x \in \ell^2_{\mathbb{C}}$. We obtain $\|T\| \leq \|a\|_{\ell^\infty_{\mathbb{C}}}$ from

$$\|Tx\|^2_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} |a_n x_n|^2 \leq \|a\|^2_{\ell^\infty_{\mathbb{C}}} \|x\|^2_{\ell^2_{\mathbb{C}}}.$$

Given any $k \in \mathbb{N}$ let $e_k = (0, \ldots, 0, 1, 0, \ldots) \in \ell^2_{\mathbb{C}}$, where the 1 is at k-th position. Then, $\|Te_k\|_{\ell^2_{\mathbb{C}}} = |a_k| = |a_k| \|e_k\|_{\ell^2_{\mathbb{C}}}$ implies $\|T\| \geq |a_k|$. Since $k \in \mathbb{N}$ is arbitrary, $\|T\| \geq \|a\|_{\ell^\infty_{\mathbb{C}}}$ follows. To conclude, $\|T\| = \|a\|_{\ell^\infty_{\mathbb{C}}}$.

(b) The adjoint operator T^* of T is given by $(T^*y)_n = \overline{a_n} y_n$ for $y \in \ell^2_{\mathbb{C}}$ because

$$\forall x, y \in \ell^2_{\mathbb{C}} \qquad (x, T^*y)_{\ell^2_{\mathbb{C}}} = (Tx, y)_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} a_n x_n \overline{y_n} = \sum_{n\in\mathbb{N}} x_n \overline{\overline{a_n} y_n}.$$

and we conclude $T = T^* \;\Leftrightarrow\; a_n = \overline{a_n} \;\; \forall n \in \mathbb{N}$.

(c) Let $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ and $e_k \in \ell^2_{\mathbb{C}}$ be as in (a). Being an orthonormal system of the Hilbert space $\ell^2_{\mathbb{C}}$, the sequence $(e_n)_{n\in\mathbb{N}}$ converges weakly to zero. If T is a compact operator, then $|a_n| = \|Te_n\|_{\ell^2_{\mathbb{C}}} \to 0$ as $n \to \infty$.

Conversely, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \infty$ and let $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ be the corresponding multiplication operator. Let $(x^{(k)})_{k\in\mathbb{N}}$ be any bounded sequence in $\ell^2_{\mathbb{C}}$ and $C > 0$ a constant such that $\|x^{(k)}\|_{\ell^2_{\mathbb{C}}} \leq C$ for every $k \in \mathbb{N}$. Since $\ell^2_{\mathbb{C}}$ is reflexive, there exists $x \in \ell^2_{\mathbb{C}}$ and an unbounded subset $\Lambda \subset \mathbb{N}$ such that $x^{(k)} \overset{\mathrm{w}}{\rightharpoonup} x$ as $\Lambda \ni k \to \infty$. In particular,

$$\lim_{\Lambda\ni k\to\infty} x^{(k)}_n = \lim_{\Lambda\ni k\to\infty} (e_n, x^{(k)})_{\ell^2_{\mathbb{C}}} = (e_n, x)_{\ell^2_{\mathbb{C}}} = x_n. \tag{$*$}$$

Moreover, since $B_C(0; \ell^2_{\mathbb{C}})$ is weakly closed, $\|x\|_{\ell^2_{\mathbb{C}}} \leq C$. Let $\varepsilon > 0$. By assumption, there exists $N \in \mathbb{N}$ such that $|a_n|^2 < \frac{\varepsilon}{4C}$ for all $n \geq N$. Assuming $a \not\equiv 0$, let $K \in \Lambda$ such that for all $\Lambda \ni k \geq K$ and each of the finitely many $n \in \{1, \ldots, N\}$ there holds

$$|x^{(k)}_n - x_n|^2 < \frac{\varepsilon}{2N\|a\|^2_{\ell^\infty_{\mathbb{C}}}}.$$

This is possible due to $(*)$. Then, for all $\Lambda \ni k \geq K$

$$\begin{aligned}\left\|Tx^{(k)} - Tx\right\|^2_{\ell^2_{\mathbb{C}}} &= \sum_{n=1}^{N} |a_n(x^{(k)}_n - x_n)|^2 + \sum_{n=N+1}^{\infty} |a_n(x^{(k)}_n - x_n)|^2 \\ &< \sum_{n=1}^{N} \frac{|a_n|^2 \varepsilon}{2N\|a\|^2_{\ell^\infty_{\mathbb{C}}}} + \frac{\varepsilon}{4C} \sum_{n\in\mathbb{N}} \Bigl(|x^{(k)}_n|^2 + |x_n|^2\Bigr) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $(Tx^{(k)})_{k\in\Lambda}$ converges in $\ell^2_{\mathbb{C}}$ which proves that T is a compact operator.

12.4. A compact operator on continuous functions

Given $a < b$, let $T\colon C^0([a,b]) \to C^0([a,b])$ be the linear operator defined by

$$(Tf)(x) = \int_a^x \frac{f(t)}{\sqrt{x-t}}\,dt.$$

(a) For every $x \in [a,b]$ and any $f \in C^0([a,b])$ there holds

$$\int_a^x \frac{1}{\sqrt{x-t}}\,dt = \Big[-2\sqrt{x-t}\Big]_{t=a}^{x} = 2\sqrt{x-a},$$
$$|(Tf)(x)| \leq \int_a^x \frac{|f(t)|}{\sqrt{x-t}}\,dt \leq 2\sqrt{x-a}\|f\|_{C^0([a,b])}.$$

Therefore, $\|Tf\|_{C^0([a,b])} \leq 2\sqrt{b-a}\|f\|_{C^0([a,b])}$ and $\|T\| \leq 2\sqrt{b-a}$. In fact, choosing a constant function f, we obtain $\|T\| = 2\sqrt{b-a}$.

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C^0([a,b])$ and let $C > 0$ be a constant such that $\|f_n\|_{C^0([a,b])} \leq C$ for all $n \in \mathbb{N}$. Then the sequence $(Tf_n)_{n\in\mathbb{N}}$ is also (uniformly) bounded in $C^0([a,b])$ since

$$\|Tf_n\|_{C^0([a,b])} \leq \|T\|\|f_n\|_{C^0([a,b])} \leq 2C\sqrt{b-a}$$

by part (a). To show equicontinuity, we consider $a \leq x \leq y \leq b$ and estimate

$$\begin{aligned}
|(Tf_n)(y) - (Tf_n)(x)| &= \Big|\int_a^y \frac{f_n(t)}{\sqrt{y-t}}\,dt - \int_a^x \frac{f_n(t)}{\sqrt{x-t}}\,dt\Big| \\
&= \Big|\int_x^y \frac{f_n(t)}{\sqrt{y-t}}\,dt - \int_a^x \Big(\frac{f_n(t)}{\sqrt{x-t}} - \frac{f_n(t)}{\sqrt{y-t}}\Big)\,dt\Big| \\
&\leq \int_x^y \frac{|f_n(t)|}{\sqrt{y-t}}\,dt + \int_a^x |f_n(t)|\Big(\frac{1}{\sqrt{x-t}} - \frac{1}{\sqrt{y-t}}\Big)\,dt \\
&\leq C\Big(\int_x^y \frac{1}{\sqrt{y-t}}\,dt + \int_a^x \Big(\frac{1}{\sqrt{x-t}} - \frac{1}{\sqrt{y-t}}\Big)\,dt\Big) \\
&\leq 2C\big(\sqrt{y-x} + \sqrt{x-a} - \sqrt{y-a} + \sqrt{y-x}\big) \\
&\leq 4C\sqrt{y-x}.
\end{aligned}$$

Hence, $|(Tf_n)(y) - (Tf_n)(x)| < \varepsilon$ whenever $|y-x| < \delta := \frac{\varepsilon^2}{16C^2}$. By the Arzelà–Ascoli theorem, $(Tf_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence, which proves that T is a compact operator.

(c) In part (a) we computed the operator norm $\|T\| = 2\sqrt{b-a}$. By definition,

$$r_T := \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}} \leq \|T\| = 2\sqrt{b-a}.$$

12.5. A multiplication operator on square-integrable functions

Given $-\infty < a \leq 0 \leq b < \infty$, let $T\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ be the linear operator defined by

$$(Tf)(x) = x^2 f(x).$$

(a) For every $f \in L^2([a,b];\mathbb{C})$, there holds

$$\begin{aligned}\|Tf\|^2_{L^2([a,b];\mathbb{C})} = \int_a^b x^4 |f(x)|^2\,dx \leq \Big(\max_{x\in[a,b]} x^4\Big)\|f\|^2_{L^2([a,b];\mathbb{C})}\\ \Rightarrow\ \|T\| \leq \max\{a^2, b^2\}.\end{aligned}$$

Suppose $b > 0$. Let $0 < \varepsilon < b$ and let $f_\varepsilon = \varepsilon^{-\frac{1}{2}}\chi_{[b-\varepsilon,b]}$, where $\chi_{[b-\varepsilon,b]}$ denotes the characteristic function of the interval $[b-\varepsilon, b] \subset [a,b]$. Then,

$$\|Tf_\varepsilon\|^2_{L^2([a,b];\mathbb{C})} = \int_{b-\varepsilon}^b x^4 |f_\varepsilon(x)|^2\,dx \geq (b-\varepsilon)^4 \|f_\varepsilon\|^2_{L^2([a,b];\mathbb{C})}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|T\| \geq b^2$. Analogously, we can prove $\|T\| \geq a^2$ under the assumption $a < 0$. In total, we obtain $\|T\| = \max\{a^2, b^2\}$.

(b) Suppose $\lambda \in \mathbb{C}$ and $f \in L^2([a,b];\mathbb{C})$ satisfy $Tf = \lambda f$. For almost every $x \in [a,b]$,

$$0 = (\lambda f - Tf)(x) = (\lambda - x^2) f(x).$$

From $\lambda - x^2 \neq 0$ for almost all $x \in [a,b]$ we conclude $f(x) = 0$ for almost all $x \in [a,b]$. Hence, $f = 0$ in $L^2([a,b];\mathbb{C})$ which proves that the operator T has no eigenvalues.

(c) In part (b) we prove that the operator $(\lambda - T)$ is injective for any $\lambda \in \mathbb{C}$. If the operator $(\lambda - T)$ were surjective, there would exist $f \in L^2([a,b];\mathbb{C})$ with $\lambda f - Tf = 1$. Then, for almost every $x \in [a,b]$,

$$1 = \lambda f(x) - Tf(x) = (\lambda - x^2) f(x) \qquad \Rightarrow\ f(x) = \frac{1}{\lambda - x^2}.$$

If $0 \leq \lambda \in \mathbb{R}$ and if $a \leq -\sqrt{\lambda}$ or $\sqrt{\lambda} \leq b$, then $f \notin L^2([a,b])$ in contradiction to our assumption because of the singularity at $x \in [a,b]$ satisfying $x^2 = \lambda$. Therefore, $(\lambda - T)$ is not surjective for $\lambda \in [0, \max\{a^2, b^2\}]$ which shows $[0, \|T\|] \subset \sigma(T)$.

If $\lambda \in \mathbb{C} \setminus [0, \|T\|]$, then the function $f\colon [a,b] \to \mathbb{C}$ with $f(x) = \frac{1}{\lambda - x^2}$ is bounded. Therefore, the map $R_\lambda\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ given by $g \mapsto gf$ is continuous. Moreover, by construction $(\lambda - T)(gf) = g$ for any $g \in L^2([a,b];\mathbb{C})$, which proves $(\lambda - T)^{-1} = R_\lambda$. To conclude, $\sigma(T) = [0, \|T\|]$.