13.1. Definitions of resolvent set

Let $A\colon D_A \subset X \to X$ be a linear operator with closed graph. The claim is that the following subsets of $\mathbb{C}$ coincide.

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is bijective, } \exists(\lambda - A)^{-1} \in L(X,X)\},$$

$$\tilde{\rho}(A) = \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is injective with dense image, } \exists(\lambda - A)^{-1} \in L(X,X)\}.$$

Let $\lambda \in \tilde{\rho}(A)$. To show $\lambda \in \rho(A)$, we need to prove that $(\lambda - A)\colon D_A \to X$ is surjective. Let $y \in X$. Since $(\lambda - A)$ has dense image, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in the image of $(\lambda - A)$ such that $\|y_n - y\|_X \to 0$ as $n \to \infty$. Let $x_n = (\lambda - A)^{-1}y_n \in D_A$. Since $(y_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in Y, and since

$$\|x_m - x_n\|_X = \|(\lambda - A)^{-1}(y_m - y_n)\|_X \leq \|(\lambda - A)^{-1}\|_{L(X,X)}\|y_m - y_n\|_X,$$

we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in X. Since X is complete, there exists a limit point $X \ni x = \lim_{n\to\infty} x_n$. Moreover,

$$Ax_n = \lambda x_n - y_n \xrightarrow{n\to\infty} \lambda x - y.$$

Since A has closed graph $x \in D_A$ with $Ax = \lambda x - y$. This implies $y = (\lambda - A)x$. Thus, $(\lambda - A)$ is surjective and $\lambda \in \rho(A)$ follows. The reverse inclusion $\rho(A) \subset \tilde{\rho}(A)$ is trivial.

13.2. Unitary operators

(a) Suppose, $T \in L(H,H)$ is an unitary operator. Then, T is invertible with inverse $T^{-1} = T^* \in L(H,H)$. In particular, T is bijective. T is also an isometry, because

$$\forall x \in H : \qquad \|Tx\|_H^2 = \langle Tx, Tx\rangle_H = \langle T^*Tx, x\rangle_H = \langle x, x\rangle_H = \|x\|_H^2.$$

Conversely, suppose, $T \in L(H,H)$ is an bijective isometry. Then, $\|Tx\|_H^2 = \|x\|_H^2$ for every $x \in H$. From the (complex) polarization identity

$$\langle x, y\rangle_H = \frac{1}{4}\Big(\|x + y\|^2 - \|x - y\|^2\Big) + \frac{i}{4}\Big(\|x + iy\|_H^2 - \|x - iy\|_H^2\Big)$$

(which is motivated by the parallelogram identity), we conclude $\langle Tx, Ty\rangle_H = \langle x, y\rangle_H$ for every $x, y \in H$. In other words, isometries preserve the scalar product. Therefore,

$$\langle T^*Tx, y\rangle_H = \langle Tx, Ty\rangle_H = \langle x, y\rangle_H$$

for every $x, y \in H$ which implies $T^*Tx = x$ for every $x \in H$. Since T is bijective, we obtain $T^* = T^{-1}$ which means that T is unitary.

(b) Let $T \in L(H,H)$ be unitary. Part (a) implies that T and $T^* = T^{-1}$ are bijective isometries. Therefore, $\|T\| = 1 = \|T^*\|$. Since the spectral radius of T is bounded from above by $\|T\| = 1$, we obtain $\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subset \rho(T)$ from Satz 6.5.3.i.

Given $\lambda \in \mathbb{C}$ with $0 \leq |\lambda| < 1$, the spectral radius of the operator (λT^*) is bounded from above by $\|\lambda T^*\| = |\lambda| < 1$. Thus, $(1 - \lambda T^*)$ is invertible on H by Satz 2.2.7. Hence, $(\lambda - T) = -T \circ (1 - \lambda T^*)$ is bijective as composition of bijective operators and we obtain $\lambda \in \rho(T)$. To conclude, $\sigma(T) \subset \mathbb{S}^1$.

13.3. Integral operators revisited

From $k(x,y) = k(y,x)$ for almost every $(x,y) \in \Omega \times \Omega$ and with the help of Fubini's theorem, we conclude that the integral operator $K\colon L^2(\Omega) \to L^2(\Omega)$ is symmetric:

$$\begin{aligned}\forall f,g \in L^2(\Omega): \quad \langle Kf, g\rangle_{L^2(\Omega)} &= \int_\Omega \Big(\int_\Omega k(x,y) f(y)\,dy\Big) g(x)\,dx \\ &= \int_\Omega f(y) \Big(\int_\Omega k(y,x) g(x)\,dx\Big)\,dy = \langle f, Kg\rangle_{L^2(\Omega)}.\end{aligned}$$

In fact, K is self-adjoint, since $D_K = L^2(\Omega) = D_{K^*}$. Therefore, the operator $A = (1-K)\colon L^2(\Omega) \to L^2(\Omega)$ is also self-adjoint (Beispiel 6.4.2.ii).

According to problem 12.1 (b), K is a compact operator, which implies that the operator $A = (1-K)$ has closed image $\operatorname{im}(A) \subset H$. According to Banach's closed range theorem, this is equivalent to $\operatorname{im}(A) = \ker(A^*)^\perp$. Since $A^* = A$, we conclude

$$A \text{ surjective} \;\Leftrightarrow\; H = \operatorname{im}(A) = \ker(A)^\perp \;\Leftrightarrow\; \ker(A) = \{0\} \;\Leftrightarrow\; A \text{ injective}.$$

13.4. Resolvents and spectral distance

(a) Given the self-adjoint operator $A \in L(H,H)$ and an element $\lambda \in \rho(A)$, the operator $(\lambda - A) \in L(H,H)$ is bijective with inverse $R_\lambda = (\lambda - A)^{-1} \in L(H,H)$. Problem 11.2 (a) then implies that R_λ^* is bijective and according to Problem 11.1 (c),

$$R_\lambda^* = \Big((\lambda - A)^{-1}\Big)^* = \Big((\lambda - A)^*\Big)^{-1} = (\overline{\lambda} - A^*)^{-1} = (\overline{\lambda} - A)^{-1} = R_{\overline{\lambda}}.$$

Alternatively, for any $x, y \in H$, we can directly compute

$$\begin{aligned}\langle x, y\rangle_H &= \langle (\lambda - A) R_\lambda x, y\rangle_H = \langle \lambda R_\lambda x, y\rangle_H - \langle A R_\lambda x, y\rangle_H \\ &= \langle R_\lambda x, \overline{\lambda} y\rangle_H - \langle R_\lambda x, Ay\rangle_H = \langle R_\lambda x, (\overline{\lambda} - A) y\rangle_H = \langle x, R_\lambda^*(\overline{\lambda} - A) y\rangle_H\end{aligned}$$

which implies $R_\lambda^*(\overline{\lambda} - A)y = y$ for any $y \in H$. According to Satz 6.5.2, resolvents commute: $R_\lambda R_{\overline{\lambda}} = R_{\overline{\lambda}} R_\lambda$. This implies that R_λ is a normal operator: $R_\lambda R_\lambda^* = R_\lambda^* R_\lambda$.

(b) Let $A, B \in L(H,H)$ be self-adjoint operators. By symmetry of the Hausdorff distance (in the sense that we can switch the roles of A and B), it suffices to prove

$$\sup_{\alpha\in\sigma(A)}\Big(\inf_{\beta\in\sigma(B)}|\alpha-\beta|\Big) \le \|A-B\|_{L(H,H)}.$$

The claim follows, if we show the following implication for any $\alpha \in \mathbb{C}$.

$$\inf_{\beta\in\sigma(B)}|\alpha-\beta| > \|A-B\|_{L(H,H)} \qquad \Rightarrow\ \alpha \in \rho(A) = \mathbb{C}\setminus\sigma(A).$$

Let $\alpha \in \mathbb{C}$ satisfy $\inf_{\beta\in\sigma(B)}|\alpha-\beta| > \|A-B\|_{L(H,H)}$. Since the claim is trivial otherwise, we may assume $\|A-B\|_{L(H,H)} > 0$. Then, α has positive distance from $\sigma(B)$ which implies $\alpha \in \rho(B)$. Hence, $(\alpha-B)^{-1}$ is well-defined and we obtain

$$(\alpha - A) = (\alpha-B) - (A-B) = \Big(1-(A-B)(\alpha-B)^{-1}\Big)(\alpha-B). \qquad (*)$$

Since $(\alpha-B)$ is bijective, it remains to prove that $\Big(1-(A-B)(\alpha-B)^{-1}\Big)$ is bijective. This follows from Satz 2.2.7 if we prove $\|(A-B)(\alpha-B)^{-1}\|_{L(H,H)} < 1$.

Consider the rational function $f_\alpha\colon \mathbb{C}\to\mathbb{C}$ given by $f_\alpha(z) = (\alpha-z)^{-1}$. By assumption,

$$\frac{1}{\|A-B\|} > \frac{1}{\inf\limits_{\beta\in\sigma(B)}|\alpha-\beta|} = \sup_{\beta\in\sigma(B)}\frac{1}{|\alpha-\beta|} = \sup\Big\{|x| \ \Big|\ x\in f_\alpha(\sigma(B))\Big\}.$$

The spectral mapping theorem (Satz 6.5.4) implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Thus,

$$\frac{1}{\|A-B\|} > \sup\Big\{|x| \ \Big|\ x\in \sigma(f_\alpha(B))\Big\} = \sup_{x\in\sigma(f_\alpha(B))}|x| = r_{f_\alpha(B)} \qquad (\dagger)$$

where we use the characterisation of spectral radius proven in Satz 6.5.3. Since $f_\alpha(B) = (\alpha-B)^{-1} =: R$ is a resolvent of B, it is a normal operator by (a). Hence,

$$\begin{aligned}
&\|Rx\|_H^2 = \langle Rx, Rx\rangle_H = \langle R^*Rx, x\rangle_H = \langle RR^*x, x\rangle_H = \langle R^*x, R^*x\rangle_H = \|R^*x\|_H^2,\\
&\|Rx\|_H^2 = \langle R^*Rx, x\rangle_H \le \|R^*Rx\|_H\|x\|_H \le \|R^*R\|\|x\|_H^2,\\
&\Rightarrow \|R\|^2 \le \|R^*R\| \le \|R^*\|\|R\| = \|R\|^2,\\
&\Rightarrow \|R\|^2 = \|R^*R\| = \sup_{\|x\|_H=1}\|R^*(Rx)\|_H = \sup_{\|x\|_H=1}\|R(Rx)\|_H = \|R^2\|.
\end{aligned}$$

(Note how the last identity makes use of the first identity.) Inductively, we obtain $\|R\|^{2^n} = \|R^{2^n}\|$ for every $n\in\mathbb{N}$ which implies $r_{f_\alpha(B)} = r_R = \|R\| = \|(\alpha-B)^{-1}\|$. Combined with estimate ($\dagger$), we obtain $\frac{1}{\|A-B\|} > \|(\alpha-B)^{-1}\|$, which yields

$$\|(A-B)(\alpha-B)^{-1}\| \le \|A-B\|\|(\alpha-B)^{-1}\| < 1$$

and proves the claim: From $(*)$ we conclude $\alpha\in\rho(A)$.

13.5. Heisenberg's uncertainty principle

Let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be densely defined, symmetric linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ such that $A(D_A) \subset D_B$ and $B(D_B) \subset D_A$.

(a) Let $x \in D_{[A,B]} := D_A \cap D_B$. Then, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big|\langle x, [A,B]x\rangle_H\Big| &\leq \Big|\langle x, A(Bx)\rangle_H\Big| + \Big|\langle x, B(Ax)\rangle_H\Big| = \Big|\langle Ax, Bx\rangle_H\Big| + \Big|\langle Bx, Ax\rangle_H\Big| \\ &\leq \|Ax\|_H\|Bx\|_H + \|Bx\|_H\|Ax\|_H = 2\|Ax\|_H\|Bx\|_H.\end{aligned}$$

(b) Since A is a symmetric operator, $\langle x, Ax\rangle_H$ is real for any $x \in D_A \subset D_{A^*}$. Indeed,

$$\langle x, Ax\rangle_H = \langle A^*x, x\rangle_H = \langle Ax, x\rangle_H = \overline{\langle x, Ax\rangle}_H.$$

Moreover, for $x \in D_A$ with $\|x\|_H = 1$, we have

$$\langle x, Ax\rangle_H^2 \leq \|x\|_H^2\|Ax\|_H^2 = \langle Ax, Ax\rangle_H$$

Therefore,

$$\mathbb{R} \ni \varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}.$$

For any $\lambda, \mu \in \mathbb{R}$, the commutators $[A,B]$ and $[A-\lambda, B-\mu]$ agree:

$$\begin{aligned}[A-\lambda, B-\mu] &= (A-\lambda)(B-\mu) - (B-\mu)(A-\lambda) \\ &= AB - \mu A - \lambda B + \lambda\mu - BA + \lambda B + \mu A - \lambda\mu = [A,B].\end{aligned}$$

Since A is symmetric and $\lambda \in \mathbb{R}$, the operator $\tilde{A} = A - \lambda$ is also symmetric on $D_{\tilde{A}} = D_A$. Moreover, for any $x \in D_A$,

$$\begin{aligned}\|\tilde{A}x\|_H^2 &= \langle \tilde{A}x, \tilde{A}x\rangle_H = \langle Ax - \lambda x, Ax - \lambda x\rangle_H \\ &= \langle Ax, Ax\rangle_H - \lambda\langle x, Ax\rangle_H - \lambda\langle Ax, x\rangle_H + \lambda^2\langle x, x\rangle_H \\ &= \langle Ax, Ax\rangle_H - 2\lambda\langle x, Ax\rangle_H + \lambda^2\langle x, x\rangle_H.\end{aligned}$$

We observe that if we choose $\lambda = \langle x, Ax\rangle_H \in \mathbb{R}$ and if $\|x\|_H = 1$, then

$$\|\tilde{A}x\|_H^2 = \langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2 = \varsigma(A,x)^2.$$

Now, let $x \in D_{[A,B]} := D_A \cap D_B$ with $\|x\|_H = 1$ be arbitrary. Since the operators $\tilde{A} := A - \langle x, Ax\rangle_H$ and $\tilde{B} := B - \langle x, Bx\rangle_H$ are symmetric, part (a) applies and yields

$$\Big|\langle x, [A,B]x\rangle_H\Big| = \Big|\langle x, [\tilde{A},\tilde{B}]x\rangle_H\Big| \leq 2\|\tilde{A}x\|_H\|\tilde{B}x\|_H = 2\varsigma(A,x)\,\varsigma(B,x).$$

(c) Suppose, $B\colon H\to H$ with finite operator-norm and $A\colon D_A\subset H\to H$ satisfy

$$[A,B]=i\,\mathrm{Id}_{D_{[A,B]}}\,.$$

By assumption, $D_{[A,B]}=D_A\cap H=D_A$ and $B(D_A)\subset D_A$. In particular, for any $n\in\mathbb{N}$ the inclusion $B^n(D_A)\subset D_A$ is satisfied, which is necessary to define $[A,B^n]$. We prove $[A,B^n]=niB^{n-1}$ by induction. For $n=1$, the claim holds by assumption. Suppose, it is true for some $n\in\mathbb{N}$. Then

$$\begin{aligned}[A,B^{n+1}]&=AB^{n+1}-B^{n+1}A=\Big(AB^n-B^nA+B^nA\Big)B-B^{n+1}A\\&=\Big([A,B^n]+B^nA\Big)B-B^{n+1}A=niB^{n-1}B+B^nAB-B^{n+1}A\\&=niB^n+B^n[A,B]=niB^n+iB^n=(n+1)iB^n.\end{aligned}$$

A consequence is that B cannot be nilpotent: If $B^n=0$ for some $n\in\mathbb{N}$, then $B^{n-1}=\frac{1}{ni}[A,B^n]=0$ which iterates to $B=0$ in contradiction to $[A,B]\neq 0$. Suppose, A has finite operator norm $\|A\|$. Then,

$$n\|B^{n-1}\|=\|[A,B^n]\|\leq\|AB^n\|+\|B^nA\|\leq 2\|A\|\|B^{n-1}\|\|B\|.$$

Since $\|B^{n-1}\|\neq 0$, we obtain $2\|A\|\geq\frac{n}{\|B\|}$ which contradicts $n\in\mathbb{N}$ being arbitrary.

(d) If $f\in C^1([0,1];\mathbb{C})$, then f' is bounded and in particular $f'\in L^2([0,1];\mathbb{C})$. The map $[0,1]\ni s\mapsto s$ is also bounded. Therefore, the linear operators

$$\begin{aligned}P\colon C^1_0([0,1];\mathbb{C})&\to L^2([0,1];\mathbb{C}), & Q\colon L^2([0,1];\mathbb{C})&\to L^2([0,1];\mathbb{C})\\ f(s)&\mapsto if'(s) & f(s)&\mapsto sf(s)\end{aligned}$$

are indeed well-defined. They are also symmetric. For Q this follows trivially from $s\in[0,1]\subset\mathbb{R}$. Given any $f,g\in D_P:=C^1_0([0,1];\mathbb{C})$, we have

$$\langle Pf,g\rangle_{L^2}=\int_0^1 if'(s)\overline{g}(s)\,ds=-\int_0^1 if(s)\overline{g}'(s)\,ds=\int_0^1 f(s)\overline{ig'(s)}\,ds=\langle f,Pg\rangle_{L^2}.$$

When integrating by parts, the boundary terms vanish due to $f(0)=0=f(1)$. Hence, $P\colon C^1_0([0,1];\mathbb{C})\to L^2([0,1];\mathbb{C})$ is symmetric (but *not* self-adjoint! see Beispiel 6.6.1).

Next, we verify that the commutator $[P,Q]$ is well-defined. Since $D_Q=L^2([0,1];\mathbb{C})$ is the whole space, the only thing to check is that $Qf\colon s\mapsto sf(s)$ is in $D_P=C^1_0([0,1];\mathbb{C})$ whenever $f\in D_{[P,Q]}=C^1_0([0,1];\mathbb{C})$. But this follows trivially from the product rule. Moreover,

$$([P,Q]f)(s)=(P(Qf))(s)-(Q(Pf))(s)=if(s)+isf'(s)-sif'(s)=if(s)$$

for almost every $s\in[0,1]$ which proves that P,Q is a Heisenberg-pair. By part (b),

$$\forall f\in C^1_0,\ \|f\|_{L^2}=1:\quad \varsigma(P,f)\,\varsigma(Q,f)\geq\tfrac12\Big|\langle f,[P,Q]f\rangle_{L^2}\Big|=\tfrac12\Big|\langle f,if\rangle_{L^2}\Big|=\tfrac12.$$