Problem 1.

(a) The following *weak separation theorem* holds: Let $(X, \|\cdot\|)$ be a normed space over the real field $\mathbb{R}$. Let $A, B \subset X$ be non-empty, convex and disjoint and let A be open. Then there exists a functional $l \in X^*$ such that

$$\sup_{a\in A} l(a) \leq \inf_{b\in B} l(b).$$

(b) Assume, for the sake of a contradiction, that the first assertion is false: then one could find a sequence (r_k) of positive real numbers, with $r_k \searrow 0$ such that $U_{r_k}(A) \cap B \neq \emptyset$ for all $k \in \mathbb{N}$. Hence we can find, for any k, points $a_k \in A$ and $b_k \in B \cap B_{r_k}(a_k)$. By sequential compactness of A (which, we recall, is equivalent to Heine-Borel compactness in the class of metric spaces) we have that, possibly extracting a subsequence which we shall not rename, $a_k \to a$ for some $a \in A$, as $k \to \infty$. However, by construction we have that $\|a_k - b_k\| < r_k$ and thus by the triangle inequality we get $\|a - b_k\| \leq \|a - a_k\| + \|a_k - b_k\|$ which implies $b_k \to a$ as $k \to \infty$. Hence, being B closed, we infer that $a \in B$ and thus $a \in A \cap B$, contrary to the assumption that the two sets are actually disjoint.

For the second assertion, observe that trivially $\sup_{a\in A} l(a) \leq \sup_{a'\in U_r(A)} l(a')$ since $A \subset U_r(A)$ and assume (again by contradiction) that the strict inequality fails, so that equality must hold i. e. $\sup_{a\in A} l(a) = \sup_{a'\in U_r(A)} l(a')$. Now, since A is compact, by the Weierstrass theorem $\sup_{a\in A} l(a)$ must be achieved at some (not necessarily unique!) maximum point $\overline{a} \in A$. It follows by the *first derivative test* that for any $v \in X$ with $\|v\| = 1$ one has that

$$\left[\frac{d}{dt}\right]_{t=0} l(\overline{a} + tv) = 0$$

which means $l(v) = 0$ for any $v \in X$ with $\|v\| = 1$ and by linearity actually $l(w) = 0$ for any $w \in X$. Thus, l would be the null functional i. e. $l = 0$ in X^*, contrary to the assumption.

(c) The following *strong separation theorem* holds: Let $(X, \|\cdot\|)$ be a normed space over the real field $\mathbb{R}$. Let $A, B \subset X$ be non-empty, convex and disjoint and assume that A is compact and B is closed. Then there exists a functional $l \in X^*$ such that

$$\sup_{a\in A} l(a) < \inf_{b\in B} l(b).$$

Let us prove this assertion using, as suggested, the results in part (a) and in part (b). Let $r > 0$ be such that $U_r(A) \cap B = \emptyset$: for this very choice of r we can apply the weak separation theorem (part (a)) to the sets $U_r(A)$ and B, thereby obtaining $l \in X^*$ such that

$$\sup_{a' \in U_r(A)} l(a') \leq \inf_{b \in B} l(b). \tag{1}$$

But on the other hand, by virtue of what we proved in part (b) we have that

$$\sup_{a \in A} l(a) < \sup_{a' \in U_r(A)} l(a') \tag{2}$$

so that combining (1) with (2) the proof is complete.

Problem 2.

(a) $A \subset X$ is of first category, if $A = \bigcup_{k \in \mathbb{N}} A_k$ with A_k nowhere dense for every $k \in \mathbb{N}$, i. e. $(\overline{A_k})^\circ = \emptyset$.

(b) For any $k \in \mathbb{N}$, let

$$A_k := \Big\{ x = (x_n)_{n \in \mathbb{N}} \in \ell^2 \;\Big|\; \sum_{n \in \mathbb{N}} n^2 |x_n|^2 \leq k \Big\}.$$

Suppose the elements $x^{(m)} \in A_k$ satisfy $x^{(m)} \to x$ in ℓ^2 as $m \to \infty$. In particular, $|x_n^{(m)} - x_n| \to 0$ as $m \to \infty$ for any $n \in \mathbb{N}$. Then, for any $N \in \mathbb{N}$

$$\sum_{n=0}^{N} n^2 |x_n|^2 = \lim_{m \to \infty} \sum_{n=0}^{N} n^2 |x_n^{(m)}|^2 \leq k.$$

Since N is arbitrary, we obtain $x \in A_k$. Hence, $A_k \subset \ell^2$ is closed. Towards a contradiction, suppose, A_k has non-empty interior. Then there exist $a = (a_n)_{n \in \mathbb{N}} \in A_k$ and some $\varepsilon > 0$ such that defining $b_n = a_n + \operatorname{sgn}(a_n) \frac{\varepsilon}{n}$ we have $(b_n)_{n \in \mathbb{N}} \in A_k$. Note that $(\frac{\varepsilon}{n})_{n \in \mathbb{N}} \in \ell^2$ with norm proportional to ε. However,

$$\sum_{n \in \mathbb{N}} n^2 |b_n|^2 \geq \sum_{n \in \mathbb{N}} \Big(n^2 a_n^2 + n \varepsilon^2 \Big) = \infty.$$

Thus, A_k is closed with empty interior, hence nowhere dense and $\mathcal{H} = \bigcup_{k \in \mathbb{N}} A_k$ is of first category.

Problem 3.

(a) *Preliminary comment: one could just present here the proof given in the lecture notes, Beispiel 5.4.1 part ii), but I shall rather present a different argument.*

We say $\ell \colon H \to \mathbb{R}$ is affine if there exist $\ell_0 \in X^*$ and $c \in \mathbb{R}$ such that $\ell(x) = \ell_0(x) + c$ for all $x \in X$. Set

$$\mathcal{A}_F := \{\ell \colon H \to \mathbb{R} \text{ affine and } \ell \leq F\}, \quad \tilde{F}(x) = \sup_{\ell \in \mathcal{A}_F} \ell(x).$$

I claim that $F(x) = \tilde{F}(x)$ which means that any convex function can be represented as supremum of the affine functions that lies below it. To check such claim, notice that by definition of $\mathcal{A}_F$ one has $F(x) \geq \tilde{F}(x)$ for all $x \in H$ and if it were $F(x_0) > \tilde{F}(x_0)$ one would reach a contradiction by invoking the weak separation theorem to $D_F := \{(x,y) \in H \times \mathbb{R} \ : y > F(x)\}$ (convex open set) and the point $(x_0, \tilde{F}(x_0)) \in H \times \mathbb{R}$, as it precisely provides an affine function $\ell \in H^*$ such that $l(x_0) > \tilde{F}(x_0)$, contradiction. Now, pick a sequence $x_k \xrightarrow{\text{w}} x$ and observe that by definition of weak convergence $\overline{\ell}(x_k) \to \overline{\ell}(x)$ for any $\overline{\ell}$ affine. We have that

$$\lim_{k\to\infty} \overline{\ell}(x_k) \leq \liminf_{k\to\infty} \sup_{\ell \in \mathcal{A}_F} \ell(x_k)$$

and hence also

$$\sup_{\ell \in \mathcal{A}_F} \lim_{k\to\infty} \ell(x_k) \leq \liminf_{k\to\infty} \sup_{\ell \in \mathcal{A}_F} \ell(x_k)$$

so that finally (by the above remark)

$$F(x) = \sup_{\ell \in \mathcal{A}_F} \ell(x) = \sup_{\ell \in \mathcal{A}_F} \lim_{k\to\infty} \ell(x_k) \leq \liminf_{k\to\infty} \sup_{\ell \in \mathcal{A}_F} \ell(x_k) = \liminf_{k\to\infty} F(x_k).$$

(b) We want to appeal to the general existence result provided by Satz 5.4.1, which can be stated (as far as we need) as follows: *Let X be a reflexive Banach space and let $T \colon X \to \mathbb{R}$ be coercive and weakly sequentially lower semicontinuous: then there exists $x_0 \in X$ such that*

$$T(x_0) = \inf_{x \in X} T(x).$$

Recalling that any Hilbert space is reflexive, it is enough to check that the functional $G \colon H \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. For the first issue, we claim that in fact

$$\lim_{\|x\|\to+\infty} \frac{F(x)}{\|x\|} = +\infty,$$

at which stage one just needs to observe that $G(x) \geq F(x) - C\|x\| = \|x\| \left(\frac{F(x)}{\|x\|} - C\right)$, where we have set $C = \sum_{i=1}^{N} \|\ell_i\|_{H^*}$, so that indeed $\lim_{\|x\|\to\infty} G(x) = +\infty$ as a result of our claim $\lim_{\|x\|\to+\infty} \frac{F(x)}{\|x\|} = +\infty$. To justify the claim, we argue as follows; let $D_F := \{(x,y) \in H \times \mathbb{R} \ : y > F(x)\}$ i. e. the *epigraph* of the function F, and let $(x_0, y_0) \in H \times \mathbb{R} \setminus D_F$ i.e. a point below the graph. By the weak separation theorem, which is applicable since $D_F \subset H \times \mathbb{R}$ is open thanks to the assumption that F is continuous, we can find $\ell \in H^*, c \in \mathbb{R}$ such that $F(x) \geq \ell(x) - c$, thus $F(x) \geq -\|\ell\|_{H^*} - c$ which implies that $F(x)/\|x\|$ is bounded from below as one lets $\|x\| \to \infty$: this implies that there cannot be any sequence (x_k) such that $\|x_k\| \to \infty$ while $F(x_k)/\|x_k\| \to -\infty$. This is precisely what one needs to gain the implication

$$\lim_{\|x\|\to+\infty} \frac{|F(x)|}{\|x\|} = +\infty \ \Rightarrow \ \lim_{\|x\|\to+\infty} \frac{F(x)}{\|x\|} = +\infty.$$

Lastly, let us prove the lower semicontinuity of G. Using part (a) (for F) we have that if $x_k \xrightarrow{w} x$ then $F(x) \leq \liminf_{k\to\infty} F(x_k)$ and for any given $\ell \in H^*$ trivially (by definition of weak convergence) $\ell(x_k) \to \ell(x)$ and thus also $|\ell(x_k)| \to |\ell(x)|$ as $k \to \infty$. Combining these two facts together gives $G(x) \leq \liminf_{k\to\infty} G(x_k)$.

Problem 4.

(a) For any $f \in L^2(\mathbb{R};\mathbb{C})$, $Tf = fg$ is measurable and

$$\|Tf\|^2_{L^2(\mathbb{R};\mathbb{C})} = \int_{\mathbb{R}} |fg|^2 \, dx \leq \|g\|^2_{L^\infty(\mathbb{R};\mathbb{C})} \int_{\mathbb{R}} |f|^2 \, dx = \|g\|^2_{L^\infty(\mathbb{R};\mathbb{C})} \|f\|^2_{L^2(\mathbb{R};\mathbb{C})}.$$

In particular, $Tf \in L^2(\mathbb{R};\mathbb{C})$ with $\|Tf\|_{L^2(\mathbb{R};\mathbb{C})} \leq \|g\|_{L^\infty(\mathbb{R};\mathbb{C})} \|f\|_{L^2(\mathbb{R};\mathbb{C})}$. As T is clearly linear, this shows that T is a continuous linear operator with $\|T\| \leq \|g\|_{L^\infty(\mathbb{R};\mathbb{C})}$.

We claim that $\|T\| \geq \|g\|_{L^\infty(\mathbb{R};\mathbb{C})}$, which will show that $\|T\| = \|g\|_{L^\infty(\mathbb{R};\mathbb{C})}$. If $\|g\|_{L^\infty(\mathbb{R};\mathbb{C})}$ vanishes then this is trivial, otherwise for any $0 < \varepsilon < \|g\|_{L^\infty(\mathbb{R};\mathbb{C})}$ the set

$$A_\varepsilon := \{x \in \mathbb{R} \ : \ |g(x)| > \|g\|_{L^\infty(\mathbb{R};\mathbb{C})} - \varepsilon\}$$

has positive measure. Assume that $|A_\varepsilon| < \infty$: since $g \neq 0$ on A_ε, we can take $f := \frac{\overline{g}}{|g|^2} \chi_{A_\varepsilon}$, which belongs to $L^2(\mathbb{R};\mathbb{C})$ since

$$\int_{\mathbb{R}} |f|^2 \, dx \leq \left(\|g\|_{L^\infty(\mathbb{R};\mathbb{C})} - \varepsilon\right)^{-2} |A_\varepsilon| < \infty$$

and moreover, being $Tf = \chi_{A_\varepsilon}$,

$$\|T\|^2 \geq \frac{\|Tf\|^2_{L^2(\mathbb{R};\mathbb{C})}}{\|f\|^2_{L^2(\mathbb{R};\mathbb{C})}} = \frac{|A_\varepsilon|}{\|f\|^2_{L^2(\mathbb{R};\mathbb{C})}} \geq \left(\|g\|_{L^\infty(\mathbb{R};\mathbb{C})} - \varepsilon\right)^2$$

(notice that f does not vanish a.e.). If instead $|A_\varepsilon| = \infty$, we choose any radius $R > 0$ such that $A_\varepsilon \cap B_R(0)$ has (finite) positive measure: this is possible because $|A_\varepsilon| = \lim_{R\to\infty} |A_\varepsilon \cap B_R(0)|$. Then we repeat the same argument with A_ε replaced by $A_\varepsilon \cap B_R(0)$, reaching again the conclusion $\|T\| \geq \|g\|_{L^\infty(\mathbb{R};\mathbb{C})} - \varepsilon$. Since ε was arbitrary, the claim follows.

(b) If $\lambda \in \mathbb{C}$ does not belong to the essential image, then there exists $\varepsilon > 0$ such that $g^{-1}(B_\varepsilon(\lambda))$ has measure zero, which means that $|g(x) - \lambda| \geq \varepsilon$ for a. e. x. Hence, the function $h(x) := (\lambda - g(x))^{-1}$ (defined a. e.) belongs to $L^\infty(\mathbb{R};\mathbb{C})$, with $\|h\|_{L^\infty(\mathbb{R};\mathbb{C})} \leq \varepsilon^{-1}$, and the corresponding multiplication operator $S\colon L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$, $Sf := fh$ satisfies

$$S(\lambda I - T) = I, \quad (\lambda I - T)S = I.$$

So $\lambda I - T$ is invertible, i.e. $\lambda \notin \sigma(T)$.

Assume instead that λ belongs to the essential image and, for any fixed $\varepsilon > 0$, let $C_\varepsilon := \{x : |g(x) - \lambda| < \varepsilon\}$, which has positive measure. As in (a), we truncate it with a ball $B_R(0)$ in the domain, in such a way that $0 < |C_\varepsilon \cap B_R(0)| < \infty$. Taking f to be the characteristic function of $C_\varepsilon \cap B_R(0)$, we get $f \in L^2(\mathbb{R};\mathbb{C})$ and

$$\frac{\|(\lambda I - T)f\|^2_{L^2(\mathbb{R};\mathbb{C})}}{\|f\|^2_{L^2(\mathbb{R};\mathbb{C})}} = \frac{\int_{C_\varepsilon \cap B_R(0)} |g(x) - \lambda|^2 \, dx}{|C_\varepsilon \cap B_R(0)|} \leq \varepsilon^2.$$

Now, if $\lambda I - T$ were invertible, we would have

$$\|f\|_{L^2(\mathbb{R};\mathbb{C})} \leq \|(\lambda I - T)^{-1}\| \|(\lambda I - T)f\|_{L^2(\mathbb{R};\mathbb{C})} \leq \varepsilon \|(\lambda I - T)^{-1}\| \|f\|_{L^2(\mathbb{R};\mathbb{C})}.$$

Thus, being $\|f\|_{L^2(\mathbb{R};\mathbb{C})} > 0$, we would get $1 \leq \varepsilon \|(\lambda I - T)^{-1}\|$, which gives a contradiction if ε is chosen small enough. So in this case $\lambda \in \sigma(T)$.

Problem 5.

Choose $H = \mathbb{R}^2$. Let $A, B \in L(\mathbb{R}^2;\mathbb{R}^2)$ be given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\|A\| = \|B\| = \|A + B\| = \|A - B\| = 1.$$

Since $2 \neq 4$, the parallelogram identity $\|A + B\|^2 + \|A - B\|^2 = 2\|A\|^2 + 2\|B\|^2$ is false in $L(\mathbb{R}^2;\mathbb{R}^2)$. Therefore, $L(\mathbb{R}^2;\mathbb{R}^2)$ is not Hilbertean.

Problem 6.

(a) $(X, \|\cdot\|_X)$ is separable if X contains a countable, dense subset. The Banach space $\left(L^\infty((0,1)), \|\cdot\|_{L^\infty((0,1))}\right)$ is not separable.

(b) $(Y, \|\cdot\|_Y)$ is reflexive, if $\mathcal{I} \colon Y \to Y^{**}$ given by $(\mathcal{I}x)(f) = f(x)$ is surjective. The Banach space $\left(L^1((0,1)), \|\cdot\|_{L^1((0,1))}\right)$ is not reflexive.

(c) Given $x \in X$, let $y_n = F_n x \in Y$. Then, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded because

$$\|F_n x\|_Y \leq \|F_n\| \|x\|_X \leq C\|x\|_X.$$

Since Y is reflexive, there exists an unbounded set $\Lambda \subset \mathbb{N}$ and some $y \in Y$ such that $y_n \xrightarrow{\mathrm{w}} y$ as $\Lambda \ni n \to \infty$ according to the Eberlein–Smulyan Theorem.

Since X is separable, there exists a dense subset $D = \{x_1, x_2, \ldots\} \subset X$. Towards a diagonal argument, let $\mathbb{N} \supset \Lambda_1 \supset \Lambda_2 \supset \ldots$ be the sets as above corresponding to the elements $x_1, x_2, \ldots \in D$. Let Λ_∞ be a diagonal sequence. Let $x \in X$ and $\ell \in Y^*$ be arbitrary. Then, for $m, n \in \Lambda_\infty$ and $k \in \mathbb{N}$, using $\|F_n\| \leq C$ we obtain

$$\begin{aligned}
|\ell(F_n x) - \ell(F_m x)| &\leq |\ell((F_n - F_m)(x - x_k))| + |\ell(F_n(x_k)) - \ell(F_m(x_k))| \\
&\leq 2C\|\ell\|_{Y^*}\|x - x_k\|_X + |\ell(F_n(x_k)) - \ell(F_m(x_k))|.
\end{aligned}$$

By density of D, the index k can be chosen such that $4C\|\ell\|_{Y^*}\|x - x_k\|_X < \varepsilon$. By the diagonal argument, $(\ell(F_n(x_k)))_{n\in\Lambda_\infty}$ is a Cauchy sequence. Hence, also $(\ell(F_n x))_{n\in\Lambda_\infty}$ is a Cauchy sequence. Since ℓ is arbitrary, $(F_n x)_{n\in\Lambda_\infty}$ converges weakly.

Problem 7.

We note preliminarily that, set $\Pi_j \in L(H, H_j)$ the orthogonal projection onto H_j, we have

$$v_j = \Pi_j(v) = \lim_{N\to\infty} \Pi_j\left(\sum_{\ell=1}^N v_\ell\right) \lim_{N\to\infty} \sum_{\ell=1}^N \Pi_j(v_\ell) \quad \forall v \in H$$

by continuity of Π_j. Moreover, being $H_k \perp H_\ell$ for $k \neq \ell$,

$$\|v\|^2 = \lim_{N\to\infty} \left\| \sum_{\ell=1}^N v_\ell \right\|^2 = \lim_{N\to\infty} \sum_{\ell=1}^N \|v_\ell\|^2 = \sum_{\ell=1}^\infty \|v_\ell\|^2.$$

($\Leftarrow$) Assume that A_c is compact. Since $H_j \neq \{0\}$ by hypothesis, for each $j \geq 1$ we can select an element $w_j \in H_j$ with $\|w_j\| = c_j$. Let us form the sequence

$$(v^{(k)})_{k=1}^\infty \subset H, \quad v^{(k)} := \sum_{\ell=1}^k w_j.$$

Note that $v^{(k)} \in A_c$ and that $v_j^{(k)} = w_j \quad \forall k \geq j$. By compactness of A_c, there exists an infinite subset $\Lambda \subset \mathbb{N}$ and a vector $v^{(\infty)} \in A_c$ such that $\lim_{\Lambda \ni k \to \infty} v^{(k)} = v^{(\infty)}$. But, by continuity of Π_j,

$$v_j^{(\infty)} = \Pi_j(v^{(\infty)}) = \lim_{\Lambda \ni k \to \infty} \Pi_j(v^{(k)}) = \lim_{\Lambda \ni k \to \infty} v_j^{(k)} = w_j$$

and so

$$\|v^{(\infty)}\|^2 = \sum_{j=1}^{\infty} \|v_j^{(\infty)}\|^2 = \sum_{j=1}^{\infty} \|w_j\|^2 = \sum_{j=1}^{\infty} c_j^2.$$

Since $\|v^{(\infty)}\|^2 < \infty$, we deduce that $c \in \ell^2$.

($\Rightarrow$) Assume that $c \in \ell^2$. Given a sequence $(v^{(k)})_{k=1}^{\infty}$ in A_c, we want to find a converging subsequence. We will reach this goal by a diagonal argument: since H_1 is finite-dimensional and $\|v_1^{(k)}\| \leq c_1$ for all k, we can find a subset $\Lambda_1 \subset \mathbb{N}$ and a vector $v_{1,\infty} \in H_1$ such that

$$\lim_{\Lambda_1 \ni k \to \infty} v_1^{(k)} = v_{1,\infty}, \quad \|v_{1,\infty}\| \leq c_1.$$

Similarly, we can find $\Lambda_2 \subset \Lambda_1$ and $v_{2,\infty} \in H_2$ such that

$$\lim_{\Lambda_2 \ni k \to \infty} v_2^{(k)} = v_{2,\infty}, \quad \|v_{2,\infty}\| \leq c_2,$$

and so on. Denoting Λ the diagonal subsequence (formed by the first element of Λ_1, the second element of Λ_2 and so on), we get

$$\lim_{\Lambda \ni k \to \infty} v_j^{(k)} = v_{j,\infty}, \quad \|v_{j,\infty}\| \leq c_j \quad \forall j \geq 1.$$

We now claim that $v^{(\infty)} := \sum_{j=1}^{\infty} v_{j,\infty}$ is well-defined, i.e. that $\lim_{N\to\infty} \sum_{j=1}^{N} v_{j,\infty}$ exists. Since H is complete, it suffices to show that we have a Cauchy sequence. Being $\sum_j c_j^2 < \infty$, by orthogonality we get

$$\left\| \sum_{j=m+1}^{n} v_{j,\infty} \right\|^2 = \sum_{j=m+1}^{n} \|v_{j,\infty}\|^2 \leq \sum_{j>m} c_j^2$$

for $m < n$, which is infinitesimal as $m \to \infty$. Note that, by uniqueness, $v_j^{(\infty)} = v_{j,\infty}$, so $v^{(\infty)} \in A_c$. We now want to show that $v^{(k)} \to v^{(\infty)}$ along the subsequence Λ. Fix any $\varepsilon > 0$ and choose $N_\varepsilon \geq 1$ such that $\sum_{j>N_\varepsilon} c_j^2 \leq \varepsilon$ (here we use $c \in \ell^2$). Then

$$\|v^{(k)} - v^{(\infty)}\|^2 = \sum_{j=1}^{\infty} \|v_j^{(k)} - v_j^{(\infty)}\|^2 \leq \sum_{j=1}^{N_\varepsilon} \|v_j^{(k)} - v_j^{(\infty)}\|^2 + \sum_{j>N_\varepsilon} (2c_j)^2,$$

where we used $\|v_j^{(k)} - v_{j,\infty}\| \leq 2c_j$. Since each term in the finite sum is infinitesimal (as $\Lambda \ni k \to \infty$), for $k \in \Lambda$ large enough we get $\|v^{(k)} - v^{(\infty)}\|^2 \leq 5\varepsilon$. Since ε was arbitrary, this proves the desired convergence.