B. Baire category

Question B.1.
Let $f \in C^0(\mathbb{R})$ such that the restriction $f|_I$ is not constant for every open, non-empty subset $I \subseteq \mathbb{R}$. Prove that $f^{-1}(\{x\})$ is nowhere dense for every $x \in \mathbb{R}$.

Question B.2.
Let $(X, \|\cdot\|_X)$ be a Banach space and let $M \subset X$ be a subset of second category. Let $(Y, \|\cdot\|_Y)$ be a normed space. Given an index set Λ, let $(A_\lambda)_{\lambda\in\Lambda} \subset L(X,Y)$ such that

$$\forall x \in M : \quad \sup_{\lambda\in\Lambda} \|A_\lambda x\|_Y < \infty.$$

Prove that

$$\sup_{\lambda\in\Lambda} \|A_\lambda\|_{L(X,Y)} < \infty.$$

Question B.3.
Prove or disprove: There are continuous functions $f_n : [0,1] \to \mathbb{R}$ for $n \in \mathbb{N}$, such that

$$[0,1] \times \mathbb{R} = \bigcup_{n\in\mathbb{N}} \Gamma(f_n),$$

where $\Gamma(f_n) = \{(x, f_n(x)) \mid x \in [0,1]\}$ is the graph of f_n.

Question B.4.
Prove that in the Banach space $C^0([0,1])$ the C^1-functions form a set of first category.

C. Continuous linear maps and fundamental theorems

Question C.1.
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces an let $T \in L(X, Y)$. Prove that the following statements are equivalent.

(i) Both, $T(X) \subset Y$ and $\ker(T) \subset X$ are topologically complemented.

(ii) There exists $S \in L(Y, X)$ satisfying the two identities

$$\begin{aligned} STS &= S, \\ TST &= T. \end{aligned}$$

Question C.2.
Let $(X, \|\cdot\|_X)$ be a normed vector space and let $M \subsetneq X$ be a linear subspace with $\overline{M} = X$. Does M have a topological complement?

Question C.3.
Let $c_c = \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \mid \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 :\ x_n = 0\}$ be equipped with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_\infty := \max_{k\in\mathbb{N}} |x_k|.$$

Consider the linear map

$$\begin{aligned} F \colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (\tfrac{1}{n} x_n)_{n\in\mathbb{N}} \end{aligned}$$

(a) Prove that F is bijective and continuous.

(b) Is the inverse F^{-1} continuous? Justify your answer. Why is this result not a contradiction to the Open Mapping Theorem (Satz 3.2.1)?

Question C.4.
Let X, Y, Z be Banach spaces, $G \in L(X, Z)$ and $H \in L(Y, Z)$. Assume that for every $x \in X$ there is a unique $y \in Y$ with $G(x) = H(y)$. Prove: The function $F : X \to Y$ with $F(x) = y$, where $y \in Y$ with $H(y) = G(x)$, is in $L(X, Y)$.

$$\begin{tikzcd} X \arrow[rr, dotted, "F"] \arrow[dr, "G"'] & & Y \arrow[dl, "H"] \\ & Z & \end{tikzcd}$$

Question C.5.

Let $(X, \|\cdot\|_X)$ be a normed space. Given $x \in X$, we define the map

$$\begin{aligned} u_x \colon \mathbb{R} &\to X \\ \lambda &\mapsto \lambda x. \end{aligned}$$

(a) Prove $u_x \in L(\mathbb{R}, X)$ and $\|u_x\|_{L(\mathbb{R},X)} = \|x\|_X$ for every $x \in X$.

(b) Prove that the following map is a bijective Isometry.

$$\begin{aligned} F \colon X &\to L(\mathbb{R}, X) \\ x &\mapsto u_x. \end{aligned}$$

Question C.6.

Consider the space $X = C^0([0,1])$ with norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$ and the operator

$$\begin{aligned} T \colon D_T := C^1([0,1]) \subset X &\to X \\ f &\mapsto f' \end{aligned}$$

(a) Prove that D_T is dense in X and that the graph of T is closed in $X \times X$.

(b) Is T continuous?

Question C.7.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $A_0, A_1 \in L(X,Y)$. For $t \in [0,1]$, let $A_t := (1-t)A_0 + tA_1$. Under the assumption

$$\exists C > 0 \quad \forall t \in [0,1] \quad \forall x \in X : \quad \|x\|_X \le C\|A_t x\|_Y,$$

prove that the statements

(i) A_0 is surjective,

(ii) A_1^* is injective with closed range,

are equivalent by using the following method: Define $I := \{t \in [0,1] \mid A_t \text{ is surjective}\}$ and prove

(a) Both, (i) and (ii) imply $I \neq \emptyset$.

(b) $I \subset [0,1]$ is both open and closed.

Combine (a) and (b) to show $I = [0,1]$ conclude the claim.

D. Dual spaces and convexity

Question D.1.
Let $(X, \|\cdot\|_X)$ be a Banach space and let $T\colon X \to X^*$ be a linear map. Prove $T \in L(X, X^*)$ under the assumption that $(Tx)(y) = (Ty)(x)$ for all $x, y \in X$.

Question D.2.
Let $(X, \|\cdot\|_X)$ be a normed vector space. Let $M \subset X$ and let $K_0, K_1 \subset M$ be extremal subsets with $K_0 \cap K_1 \neq \emptyset$. Prove that $K_0 \cap K_1 \subset M$ is an extremal subset.

Question D.3.
If M is a set, then $\operatorname{ep}(M)$ denotes the set of all extremal points of M. Let $A \neq \emptyset$ be a set and let $B \subset A$ be an extremal subset of A. Prove that

$$\operatorname{ep}(B) = B \cap \operatorname{ep}(A).$$

Question D.4.
Find all extremal points in the closed unit ball $\{x \in X \mid \|x\|_X \leq 1\}$ in the case of

(a) $X = C^0([0,1] \cup [2,3])$ with norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1]\cup[2,3])}$,

(b) $X = L^1([0,1])$ with norm $\|\cdot\|_X = \|\cdot\|_{L^1([0,1])}$.

W. Weak topology

Question W.1.
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in L(X, Y)$.

(a) Prove that T is weakly-weakly sequentially continuous which means that for every sequence $(x_k)_{k\in\mathbb{N}}$ in X with $x_k \xrightarrow{\mathrm{w}} x$ as $k \to \infty$ we have $Tx_k \xrightarrow{\mathrm{w}} Tx$ as $k \to \infty$.

(b) Prove that if X is reflexive, then $T(\overline{B_1(0)})$ is closed.

Question W.2.
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $T\colon X \to Y$ be a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$ Prove that the graph $\Gamma \subset X \times Y$ of T is weakly sequentially closed.

Question W.3.
Let $(X, \|\cdot\|_X)$ be a normed vector space and let $T \in L(X, X)$. Prove that $\ker(T)$ is weakly sequentially closed.

Question W.4.
Let $(X, \|\cdot\|_X)$ be a Banach space. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X and $x \in X$. Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in X^* and $y \in X^*$. Under the assumptions that

$$\|x_k - x\|_X \to 0 \quad \text{and} \quad y_k \xrightarrow{\mathrm{w}^*} y$$

as $k \to \infty$, prove that $\lim\limits_{k\to\infty}\Big(y_k(x_k)\Big) = y(x)$.

Question W.5.
Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $L^1([0,1])$ such that the limit

$$\lim_{k\to\infty} \int_0^1 f_k(t) g(t)\,\mathrm{d}t$$

exists for all $g \in C^0([0,1])$. Prove that there exists a constant $C > 0$ such that

$$\forall g \in C^0([0,1]) : \qquad \sup_{k\in\mathbb{N}} \left| \int_0^1 f_k(t) g(t)\,\mathrm{d}t \right| \leq C \|g\|_{C^0([0,1])}.$$

Question W.6.
For any $n \in \mathbb{N}$ let the functional $T_n\colon L^\infty((0,\infty)) \to \mathbb{R}$ be defined by

$$T_n f = n\bigg(\int_0^1 x^n f(x)\,\mathrm{d}x + \int_1^\infty \mathrm{e}^{-nx} f(x)\,\mathrm{d}x\bigg).$$

(a) Compute the operator norm $\|T_n\|$ for any $n \in \mathbb{N}$.

(b) Determine (if it exists) a functional T such that $T_n \xrightarrow{\mathrm{w}^*} T$ as $n \to \infty$.

S. Spectral theory

Question S.1.
Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{C}$ and let $T \in L(X, X)$ such that $T \circ T = T$. Determine the spectrum $\sigma(T)$.

Question S.2.
(a) Consider the operator

$$\begin{aligned} T\colon \ell^2 &\to \ell^2 \\ (\alpha_1, \alpha_2, \alpha_3, \ldots) &\mapsto (\alpha_2, \alpha_3, \alpha_4, \ldots). \end{aligned}$$

Prove that $\sigma(T) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$.

(b) Consider the operator

$$\begin{aligned} S\colon \ell^2 &\to \ell^2 \\ (\alpha_1, \alpha_2, \alpha_3 \ldots) &\mapsto (\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_3 + \alpha_4 + \alpha_5, \ldots). \end{aligned}$$

Determine the spectrum $\sigma(S)$ using (a).

Question S.3.
Given $f \in X := C^0([0,1], \mathbb{C})$ we define the map

$$\begin{aligned} M_f\colon X &\to X \\ g &\mapsto fg \end{aligned}$$

(a) Prove that M_f has spectrum $\sigma(M_f) = f([0,1])$.

(b) Prove that M_f has point spectrum $\sigma_p(M_f) = \{\lambda \in \mathbb{C} \mid (f^{-1}(\{\lambda\}))^\circ \neq \emptyset\}$.

(c) Prove that the eigenspace of every eigenvalue of M_f is infinite-dimensional.

Question S.4.
Let $g \in L^\infty(\mathbb{R})$ and consider the operator $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $Tf = fg$.

(a) Compute the norm of T.

(b) Compute the real spectrum $\sigma_{\mathbb{R}}(T)$ of T.

H. Miscellaneous questions on Hilbert spaces

Question H.1.
Let $(H, (\cdot,\cdot)_H)$ be a separable Hilbert space and let $(e_k)_{k\in\mathbb{N}}$ be an Hilbertean basis. Let $x, y \in H$. Give a complete proof of the identity

$$(x, y) = \sum_{k=1}^{\infty}(x, e_k)(y, e_k)$$

Question H.2.
Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space. Let $Y \subset H$ be a closed subspace and let

$$\begin{aligned}\pi_Y\colon H = Y \oplus Y^{\perp} &\to Y\\ x = x^{\parallel} + x^{\perp} &\mapsto x^{\parallel}\end{aligned}$$

be the orthogonal projection onto Y.

(a) Is π_Y symmetric?

(b) Is π_Y self-adjoint?

Question H.3.
Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$. Prove that weak convergence $x_n \xrightarrow{\text{w}} y$ to an element $y \in H$ with norm $\|y\|_H = 1$ implies norm convergence $\|x_n - y\|_H \to 0$ as $n \to \infty$.

Question H.4.
Consider the Hilbert space $X := (\ell^2, (\cdot,\cdot)_{\ell^2})$. Let $e_k := (0, \ldots, 0, 1, 0, \ldots) \in X$, where the 1 is at k-th position. Is the set

$$K = \left\{ a = \sum_{i=1}^{\infty} a_k e_k \in X \;\middle|\; \forall k \in \mathbb{N}:\ |a_k| \leq \frac{1}{k} \right\}$$

(a) convex?

(b) weakly sequentially compact, i. e. does every sequence in K have a subsequence which converges weakly in K?

(c) compact?

Question H.5.
Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space. Given $1 < p < \infty$, let $F\colon H \to \mathbb{R}$ be defined by $F(u) = \frac{1}{p}\|u\|_H^p$.

(a) Prove that for every $v \in H$ the supremum

$$G(v) := \sup_{u\in H}\Big((u,v)_H - F(u)\Big)$$

is attained at some $u = u(v) \in H$.

(b) Find an equation for $u = u(v)$ and determine $G(v)$.