WHAT IS FUNCTIONAL ANALYSIS?

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This informal note serves as a general presentation for the course *Functional Analysis*.

1. Introduction

One of the most important concepts you have encountered in your studies so far is certainly that of **vector space**: roughly speaking, this is a set V endowed with two operations

$$+ : V \times V \to V \qquad (v_1, v_2) \mapsto v_1 + v_2$$

called **sum of two vectors**, and

$$\cdot : \mathbb{K} \times V \to V \qquad (\lambda, v) \mapsto \lambda \cdot v$$

called **multiplication of a vector by a scalar in the base field** $\mathbb{K}$, that satisfy a list of very reasonable axioms. Notice that is a purely algebraic notion (like those of group, ring, A-module etc ...), and a priori no topology comes into play.

Now, a key concept in the study of vector spaces is that of **basis**. In your courses you studied that $\{v_i\}_{i\in I}$ is a basis for V if it is a linearly independent family which generates any vector $v \in V$ in the sense that any given vector v can be expressed as a finite linear combination

$$v = \lambda_1 v_{i_1} + \lambda_2 v_{i_2} + \ldots + \lambda_k v_{i_k}$$

with coefficients $\lambda_{i_1}, \ldots, \lambda_{i_k} \in \mathbb{K}$. Now, the index set I parametrizing the basis can be either finite or infinite and first-year linear algebra courses are indeed concerned with the study of those vector spaces that do admit a basis consisting of finitely many elements. In that case, one calls **dimension** of the vector space the number of elements in a basis, and it is an important fact that this number is an invariant of the vector space (namely: it does not depend on the choice of the basis in question). Vector spaces admitting a basis of finitely many elements are then called finite-dimensional.

And here we come to the point of this course: not all vector spaces are finite-dimensional and in fact many of the most interesting classes of vector spaces in mathematics do not fall in such category. When a vector space V does not admit a basis of finitely many elements we shall say that it is an **infinite-dimensional vector space**. A first example of an infinite dimensional vector space you have already seen is given by the algebra of polynomials with real coefficients, which is usually denoted by $\mathbb{R}[x]$. How do you check such claim? A standard approach with these sorts of problems is to argue by contradiction. Let us then assume that $\mathbb{R}[x]$ has a finite basis, and let n be its cardinality. Then (by first-year linear algebra) any collection of $n+1$ elements in $\mathbb{R}[x]$ should be linearly dependent, meaning that it should satisfy a linear equation with coefficients in $\mathbb{R}$. In particular, such assertion should hold true for the family $\{1, x, x^2, \ldots, x^n\}$ so one can find real numbers $\lambda_0, \ldots, \lambda_n$, not all zero, such that

$$\sum_{i=0}^{n} \lambda_i x^i = 0,$$

identically for all $x \in \mathbb{R}$. We know that this contradicts the fundamental theorem of algebra, since any non-zero polynomial of degree n can have at most n distinct roots in $\mathbb{C}$ (hence in $\mathbb{R}$), but let us pretend to ignore this fact and provide a direct argument instead. Here it is. If we evaluate this equation at $x_0 = 0$ we get that $\lambda_0 = 0$. We can now bring a **morphism** into play and consider the **linear map** $d/dx : \mathbb{R}[x] \to \mathbb{R}[x]$, the usual derivative operator. If we differentiate the equation we find

$$\sum_{i=1}^{n+1} i\lambda_i x^{i-1} = 0$$

and thus, evaluating again at $x_0 = 0$ we find $\lambda_1 = 0$. Proceeding by induction, after exactly n steps we get to the contradictory conclusion

$$\lambda_0 = \lambda_1 = \ldots \lambda_n = 0$$

and thus we have proven the claim that $\mathbb{R}[x]$ is not a finite-dimensional vector space.

But there are many more! In fact, you have already studied many examples of vector spaces (over $\mathbb{R}$) that do not admit a finite basis in the sense above, the very simplest ones being provided by $L^p(0,1)$ for any $p \in [1, \infty]$. To clarify the point and be more concrete, let us study the explicit case of $L^2(0,1)$. Is it possible to find a finite, but possibly large, number of functions $\psi_1, \ldots, \psi_n$ whose collection is a basis for such vector space? Again, let us argue by contradiction. Well, if that were the case, any set of $n+1$ functions in $L^2(0,1)$ should be linearly dependent, that is to say for every $\varphi_1, \ldots, \varphi_{n+1}$ one could find real numbers $\lambda_1, \ldots, \lambda_{n+1}$ (not all zero!) such that the linear equation

$$\sum_{i=1}^{n+1} \lambda_i \varphi_i = 0$$

is satisfied. So, as a special case, such assertion should hold true for the family $\sin(k\pi x)$ as we let $k = 1, 2, \ldots, n+1$. At this stage we proceed in the argument by bringing a new **structure** into play, the standard **scalar product** $\langle \cdot, \cdot \rangle : L^2(0,1) \times L^2(0,1) \to \mathbb{R}$ defined by

$$\langle \vartheta_1, \vartheta_2 \rangle = \int_0^1 \vartheta_1(x)\vartheta_2(x)\, dx.$$

A simple calculation shows that any two functions in the family above are orthogonal, namely

$$\langle \sin(k_1\pi x), \sin(k_2\pi x) \rangle = 0 \quad \text{if } k_1 \neq k_2.$$

Hence, if we had

$$\sum_{i=1}^{n+1} \lambda_i \sin(i\pi x) = 0$$

then also, for any fixed $k = 1, 2, \ldots, n+1$

$$0 = \sum_{i=1}^{n+1} \int_0^1 \lambda_i \sin(i\pi x) \sin(k\pi x)\, dx = \frac{\lambda_k}{2}$$

so we conclude that $\lambda_k = 0$ for every $k = 1, 2, \ldots, n+1$, which is a contradiction.

The argument above shows that $L^2(0,1)$ is such a space, but with a slightly more abstract argument one can easily show that in fact $L^p(\Omega)$ is never finite-dimensional, whatever choice of the exponent p and of the open set $\Omega \subset \mathbb{R}^n$.

These examples being given, functional analysis is often defined as **the branch of Mathematics that is concerned with the study of infinite-dimensional vector spaces**. A less satisfactory, but perhaps more vivid definition, would be that functional analysis is the study of those **spaces whose points are functions** in the sense exemplified by, say, the function $\sin(\pi x)$ as a point in $L^2(0,1)$. Obviously, when we say that we tacitly mean that not only vector spaces comes into play, but also linear maps among them (which do constitute further linear spaces, in fact). In the language of category theory, the study is not only about **objects** (vector spaces) but also about associated **morphisms** (linear maps).

There are two remarks that I would like to add to this definition as it stands. The first is that definition was essentially conceived and adopted around the middle of the last century, in an era where a lot of emphasis in Analysis was put on linear structures, motivated by the development of the theory of distributions by Laurent Schwartz on the one hand, and by the strive for a general theory of linear partial differential equations by Lars Hörmander on the other. Nowadays, what they used to call functional analysis is something that I would rather call *linear* functional analysis, to distinguish it from the study of infinite-dimensional manifolds (which is instead the study of those curved spaces, like Banach or Hilbert manifolds, that are locally modelled by Banach or Hilbert spaces in the same way that a sphere is locally modelled on $\mathbb{R}^2$). In this sense, one may keep in mind the naive proportion

$$\begin{aligned}&(\text{linear algebra}) : (\text{differential geometry}) \\ &\qquad = (\text{linear functional analysis}) : (\text{nonlinear functional analysis}).\end{aligned}$$

The second comment is that a peculiar trait of functional analysis is the *abstract perspective*, that is to say a constant attempt not to study functional spaces one at a time, but rather to isolate certain structures/properties and to prove theorems that apply uniformly to all spaces that do have those structures and/or that do satisfy those properties. Of course, this requires some serious efforts when first learning the subject, but provides those who are not afraid of the journey with tools that have proven to be incredibly powerful and effective not only in shaping the landscape of contemporary Mathematics, but also in a variety of applications. Functional analysis provides the natural language of **quantum mechanics**, hence it is a key component of the background of any theoretical physicist or chemist, but it is also the natural framework for the study of **partial differential equations**, which makes this subject essential in a variety of contexts ranging from fluid-dynamics to general relativity.

2. A roadmap

I will now try to describe the contents of the course, and to briefly outline what we will cover during the first semester. Finite-dimensional vector spaces are always, and somehow trivially (by their very definition), isomorphic to $\mathbb{R}^n$ (for some $n \geq 0$) and as a such they naturally inherit various sorts of structures. Let us recall that $\mathbb{R}^n$ naturally comes with its Euclidean scalar product

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and this induces (in turn, and as a logical cascade)

- a **norm** by means of the formula
$$\|x\| := \sqrt{x \cdot x}$$
- a **distance** by means of the formula
$$d(x, y) = \|x - y\|$$
- a **topology** by means of the formula
$$U \subset \mathbb{R}^n \text{ is open} \Leftrightarrow \ \forall x \in U \ \exists \ r \in \mathbb{R}_{>0} \mid B_r(x) := \{y \in \mathbb{R}^n \mid d(x, y) < r\} \subset U.$$

It is also worth recalling that in finite dimension all norms are equivalent, in the sense that given any two $\| \cdot \|, \| \cdot \|'$ one can find a positive constant $C > 0$ such that

$$C^{-1}\|x\| \leq \|x\|' \leq C\|x\| \ \ \forall x \in \mathbb{R}^n$$

so that one might be tempted to say (in informal but effective terms) that only one normed structure actually exists, namely that induced by the Euclidean scalar product. Notice also that the equivalence of all norms implies at once the equivalence of any two distances induced by norms and hence the fact that any topology induced by a norm is equivalent to the Euclidean topology, in the sense that the two topologies share the same open sets.

On the contrary, infinite-dimensional vector spaces do not directly come with a scalar product, in fact they do not a priori naturally come with any of the structures above. Perhaps more significantly, in such context the structures above are very well-distinct in the sense that, for instance, there are typically lots of norms that are not induced by inner products and so on. As a result, when endowing a vector space V with one of the structures above (scalar product, norm, distance, topology), we are truly looking at different classes of vector spaces, the sole common requirement being **completeness** of the space in question, which is necessary for the most basic operations in Analysis. For instance, we shall define a **Hilbert space** (resp. a **Banach space**) to be a vector space with a scalar product (resp. a norm), such that the associated topology is complete. Associated to such a **hierarchy of structures** there is a natural chain of inclusions

$$\begin{aligned}&\{\text{Hilbert spaces}\} \subset \{\text{Banach spaces}\} \\ &\qquad \subset \{\text{(complete) linear metric spaces}\} \subset \{\text{(complete) topological vector spaces}\}.\end{aligned}$$

and one of the general scopes of functional analysis is to study the specific properties of any of these classes of spaces. As a good, vague principle to keep in mind: the stronger the structure, the richer the theory.

A wild approach to the subject could be to divide the basic theorems in functional analysis into two basic categories:

(1) those asserting that for some of the classes above (say e. g. Banach spaces) and with respect to a given property *things work like in finite dimension*, but perhaps with much more effort;
(2) those asserting that for some of the classes above (say e. g. Banach spaces) and with respect to a given property *things work very differently from the finite-dimensional scenario*.

An example from the first category is the study of **duality** for Hilbert spaces: the **Riesz representation theorem** for Hilbert spaces, asserting that any continuous linear functional equals the scalar product with a vector (so that a scalar product induces an isomorphim $V \simeq V^*$ between a space and its dual). This is a seemingly innocent, but in fact very powerful tool in proving the existence of weak solutions for various classes of linear partial differential equations, as we shall see.

An example from the second category is the study of **compactness** for Banach spaces: we will see that the closed unit ball in a Banach space is compact (for the normed topology) if and only if the space in question is finite-dimensional, with the dramatic implication that all infinite-dimensional Banach spaces (like e. g. $L^p(0,1)$ for all p) have a closed unit ball that is *not* compact. This turns out to be a very serious issue, which motivated the introduction of key notions like those of weak (and weak*) topology.

Once these basic topological phenomena are understood, one can turn to what is probably the single most important scope of functional analysis, which is the investigation of the solvability of linear equations of the form

$$Tv = w$$

where $T : V \to W$ is a continuous linear map, acting between vector spaces V, W belonging to the one of the classes described above. A good example to keep in mind, probably familiar to most of you, would be the study of the Laplace equation with Dirichlet boundary conditions

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $f \in L^2(\Omega)$ and Ω denotes a compact, regular domain in $\mathbb{R}^n$, like the unit ball centered at the origin. In this respect, the first goal of our investigation will be to introduce a suitable functional framework for the problem in question, which allows to pose the (unique) solvability of the equation above in the equivalent terms of showing that the differential operator $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is a **linear isomorphism**. Very concretely, this relies on defining the Sobolev spaces $H^k(\Omega)$ and their subspaces $H^k_0(\Omega)$, which offer a convenient way to encode the boundary condition that $u = 0$ on $\partial\Omega$. Once this abstract perspective is gained, our second scope will be to develop those tools that are needed to check (with least possible effort, and in greatest possible generality) that the mapping in question is indeed an isomorphism. A very important result in the study of linear maps will be the **spectral theorem**, which precisely asserts that certain classes of linear operators (namely those that are **compact and self-adjoint**) among Hilbert spaces can be diagonalized. Needless to say, this results (and its very many extensions) play a crucial role in the foundations of modern physics, and in fact in several other applications.

3. Some advice

In this final section, I would like to try to give you some advice aimed at making your study of the topics we will cover effective and possibly pleasant. In this class you will come in contact with some beautiful mathematics and the tools you will acquire will be a central part of your scientific background. For many of you this will be the your first journey in the landscape of twentieth century mathematics and you will face its most challenging sides for the first time: probably you will find some of the theorems that we will see (and the corresponding proofs!) quite abstract, but you should **never feel discouraged** as that is the natural price to pay to get closer to the border of human knowledge.

In class. Coming to class might be very helpful for your success in understanding the content of this course: we will do our very best in order to make your efforts valuable. The lectures should guide you in getting some feeling and intuition for the topics we will cover and, more practically, should guide your personal study: they will enable you to understand what are the conceptual keys of each chapter of the textbook, the points you have to study most carefully. While in class you should try to follow the lectures very carefully and take some notes. With this respect, everybody has his/her own style and there is no general recipe that works for everyone. I suggest to avoid just copying what the instructor writes on the blackboard and there are two main reasons for this. First: often what the instructor says is more helpful than what he actually writes down (at least in conveying ideas). Second: your work in class should be **active** and you should do a sort of minimal (since it is necessarily real-time) re-elaboration of the material. Concretely: write down concepts in your own words, add an extra picture, use different colours, make an asterisk next to the point you do not understand. Another fundamental principle is: ask questions! Do not be scared to stop the instructor when something is not clear, since often your doubts are common to other students as well. Moreover, asking questions might also help the instructor to select the points that should be reviewed and/or recalled in the course.

Your study. Making the most profit out of the lectures might help you a lot, but still you will have to spend some time studying the material covered in class and doing your homework. Notice that I did not simply say ‘doing your homework’ since I believe that any effective study session should always begin with a brief review of the theory, both on your notes and on the textbook. While studying try to have a critical/skeptical attitude: ask yourself simple questions (and try to answer them!), play with the tools you learn, draw pictures to help geometric intuition. When reading a proof check carefully the point(s) where each assumption (hypothesis) is needed. Ideally (but this might be a bit hard at the beginning) you should try to prove a theorem by yourself before actually studying it: even if you do not succeed, your study will be much easier and faster then and you will remember the proof much better. Some of the problems you will do are quite standard, but others might require a bit of creativity and some original idea. You should not be scared by these problems, instead you should progressively acquire your skills in attacking them. Be at the same time flexible and stubborn. Flexible in the sense that you should consider several different approaches before committing to one. Stubborn in the sense that you should not give up if your approach does not work immediately. Remember that if you can solve a problem immediately, then there is not much gain out of that solution. You should never be frustrated: be optimistic and **enjoy** the process of learning!

Writing a mathematical argument. A relevant fraction of the time you will spend for this class will be in writing down solutions of problems. Try to take this seriously: working in groups is great and may help in the learning process, yet you should then always write down your own solution. Your solutions should be neat and complete. 'Neat' means well-structured, not only esthetically, but also logically. 'Complete' means that a solution is not really about giving an answer (like: yes or 27, which can be derived by infinitely many correct and infinitely many wrong procedures), but to produce an argument that is solid, irreproachable and explains why one is led to a certain conclusion, and nowhere else.

Time managment. This is a very important point: studying mathematics is effective only if it is a regular activity, by which I mean that you should build your own weekly routine, like an avid athlete would do when preparing for a race. You should try to fix every day the topics covered in class that day. Do not postpone! Studying right before the exam is both (almost) useless and frustrating! With little but regular effort you will not only pass this exam with an excellent grade, but also (and most importantly) you will learn something.

Resources. First of all, I do encourage active and regular participation to our weekly problem sessions: they will give you the opportunity to review the topics in smaller groups, to discuss problems and see some of them solved in great detail.

Secondly, if you do not feel comfortable with some topic, or if you simply wish to discuss something with us you should definitely come to office hours. When you do so, my advice is to prepare quite precise questions so that you can come back home with precise answers. Things have been arranged so to have four office hours every week of the semester: that seems to be quite an experiment here at ETH, and my hope is that such choice will turn out to be successful. This will depend on each of you, on your attitude towards this course and to your willingness to take on the challenge you have in front of you. Besides the hours offered by the assistants, please feel free to contact me whenever you want. You can either send me an email to arrange a meeting or simply stop by. Whenever you see my door open, please do not hesitate: come in and feel welcome, for **you** are the reason why I am here.

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