Exercise Sheet 1

Please hand in your solutions by September 25, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Show that a subset $M \subset \mathbb{R}^k$ is a $0-$dimensional submanifold if and only if M is discrete, i.e. for every $p \in M$ there is an open set $U \subset \mathbb{R}^k$ such that $U \cap M = \{p\}$.

b) Show that a subset $M \subset \mathbb{R}^m$ is a $m-$dimensional submanifold if and only if M is open.

c) If $M_i \subset \mathbb{R}^{k_i}$ is a m_i-manifold for $i = 1, 2$ show that $M_1 \times M_2$ is a $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{k_1+k_2}$. Prove by induction that the $n-$torus $\mathbb{T}^n$ is a smooth submanifold of $\mathbb{C}^n$.

2. Consider the subset $L = \{(x, y) \in \mathbb{R}^2 : xy = 0\} \subset \mathbb{R}^2$.

a) Prove that L is not a $0-$dimensional submanifold of $\mathbb{R}^2$.

b) Prove that L is not a $1-$dimensional submanifold of $\mathbb{R}^2$.

c) Prove that L is not a $2-$dimensional submanifold of $\mathbb{R}^2$.

Hint: Use Exercise 1 in a) and c). Use Theorem 1 from the lecture in b).
Find another proof in b) using the connected components of the complement of a point.

3. **a)** Show that $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$ is a n-dimensional submanifold of $\mathbb{R}^{n+1}$ by using Theorem 1.

b) Write down a small collection of charts $\varphi_i : V_i := U_i \cap S^n \to \Omega_i \subset \mathbb{R}^n$ such that $\bigcup_{1\le i\le N} V_i = S^n$. What is the smallest N?

Hint: Try stereographic projections from North and South poles.

c) For a fixed pair (i, j) with $i \neq j$ calculate the *change of chart*

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j \cap S^n) \to \varphi_j(U_i \cap U_j \cap S^n).$$

4. **a)** Show that the general linear group

$$GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

b) Show that the special linear group

$$SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

c) Show that the orthogonal group[1]

$$O(n) = \{A \in \mathbb{R}^{n\times n} : A^\top A = \mathbb{1}\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

d) Show that the symplectic group[2]

$$Sp(2n,\mathbb{R}) = \{A \in \mathbb{R}^{2n\times 2n} : A^\top J_0 A = J_0\} \text{ with } J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

Hint: Use Exercise 1 for a). Use Theorem 1 with appropriate target spaces (a subset of all matrices) in b), c) and d). Jacobi's formula is handy in b).

5. Let $M \subset \mathbb{R}^k$ be a submanifold of $\mathbb{R}^k$ and define the distance function d_0 on M by $d_0(p,q) = |p-q|$ for $p, q \in M$. Show that the topology on M induced by d_0 is the same topology as the subspace topology induced on M as a subset of $\mathbb{R}^n$. Recall that the subspace topology is defined by $V \subset M$ is open exactly if there is an open set $U \subset \mathbb{R}^k$ such that $V = U \cap M$.

6. **a)** Prove that path connected topological spaces are connected.

b) Prove that any connected submanifold $M \subset \mathbb{R}^k$ is also path connected.

Hint: Prove first that any submanifold is locally path connected. I.e. for every $p \in M$ there is an open set $p \in V \subset M$ such that V is path connected.

[1] This group is important for Riemannian geometry.
[2] This group is important for symplectic geometry.