Exercise Sheet 2

Please hand in your solutions by October 2, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (kinetic plus potential energy) by $H(x, y) := \frac{1}{2}|y|^2 + V(x)$. Prove that c is a regular value of H if and only if it is a regular value of V.

2. **a)** Given an open set $U \subset \mathbb{R}^k$. What are its tangent spaces?

b) Given two submanifolds M_1, M_2, what are the tangent spaces of the product manifold $M_1 \times M_2$?

c) Given $U \subset \mathbb{R}^k$ open and a smooth function $f : U \to \mathbb{R}^\ell$. What are the tangent spaces of $\text{graph}(f) := \{(x, y) \in \mathbb{R}^{k+\ell} : x \in U, y = f(x)\}$?

d) What are the tangent space of the sphere S^n ?

e) What is the tangent space $T_{\mathbb{1}}SL(n, \mathbb{R})$?

f) What is the tangent space $T_{\mathbb{1}}O(n)$?

Hint: For a), b), d), e), look back at ExSheet 1 Ex 1 c), Ex 3 and Ex 4 b), c). For c), example 1.2.3. of the lecture notes might come in handy. Use the Theorem from the lecture on tangent spaces in the different settings.

3. Show that $K := \{(x, y, z) \in \mathbb{C}^3 : x^4 + y^4 + z^4 + 1 = 0\} \subset \mathbb{C}^3 \cong \mathbb{R}^6$ is a smooth manifold and write down its tangent spaces. What is the dimension of K?

K is called a K3 surface and is an important object of study in complex algebraic geometry.

Hint: Get real, and you shall be caught up in a huge mess. Continue to use complex variables. Just use Theorem 1 and remember holomorphic means complex linear differential.

4. [1]Let $M := \{(x^2, y^2, z^2, xz, yz, xy) \in \mathbb{R}^6 : (x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$.

a) Show that M is a submanifold of $\mathbb{R}^6$. What is the dimension of M?

b) What are the tangent spaces of M?

c)* Prove that M is diffeomorphic to the projective plane $\mathbb{R}P^2 \cong S^2/\{\pm 1\}$.

Hint: Look at the smooth map $f : S^2 \to \mathbb{R}^6 : (x, y, z) \to (x^2, y^2, z^2, xz, yz, xy)$. The image of f is M and f is not injective. In fact, f is quite special and has to do with c).

[1]Exercises with * are reserved for the interested student.

5. Decide for the following maps whether...

i) ...they are injective.

ii) ...they are an immersion.

iii) ...they are proper.

iv) ...their image is a submanifold.

a) $f : S^n \subset \mathbb{R}^{n+1} \to \mathbb{R}^{2n} : x \mapsto (x_1, \ldots, x_n, x_{n+1}x_1, \ldots, x_{n+1}x_n)$.

b) $g : \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right) \to \mathbb{R}^2 : t \mapsto (\cos(t), \cos(t)\sin(t))$.

c) $h : (-\pi, \pi) \to \mathbb{R}^2 : t \mapsto (\cos(t), \sin(t))$.

d) $i : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (\cos(t^5), \sin(t^5))$.

e) $j : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t^2, t^3)$.

f) $k : \mathbb{R} \to \mathbb{R} : t \mapsto t^3$.

6. Recall that we call a map between manifolds $f : M \subset \mathbb{R}^k \to N \subset \mathbb{R}^\ell$ proper, if $f^{-1}(K)$ is compact, whenever $K \subset f(M)$ is compact.

a) Show that the following are equivalent for continuous maps:

- $f^{-1}(K)$ is compact, for all $K \subset N$ compact.
- f is proper and $f(M)$ is closed.

b) Let M be an open subset of a manifold $N \subset \mathbb{R}^k$. Prove that the inclusion map $i : M \to N$ is proper.