Exercise Sheet 3

Please hand in your solutions by October 9, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Calculate the flow of the following vector fields and find all their zeros.

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$

c) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, -y)$

d) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y, -x)$

e) $f : S^2 \to \mathbb{R}^3$, $f(p) = e_3 \times p$ with $e_3 = (0, 0, 1)$

f) $f : S^2 \to \mathbb{R}^3$, $f(p) = (e_3 \times p) \times p$ with $e_3 = (0, 0, 1)$

g) $f : S^3 \to \mathbb{R}^4$, $f(x_1, y_1, x_2, y_2) = (-y_1, x_1, -y_2, x_2)$

2. The goal of this exercise is to prove the following.

$$\text{Every vector field on a compact manifold is complete.} \tag{1}$$

Thus, let $M \subset \mathbb{R}^k$ be a compact manifold and let $X : M \to \mathbb{R}^k$ be a vector field on M. We need to prove the following steps.

a) Use Theorem 8 to prove that $U_\epsilon := \{p \in M : [-\epsilon, \epsilon] \subset I(p)\}$ is open for all $\epsilon > 0$.

b) Use Theorem 7 to prove that there is $\epsilon_0 > 0$ such that $M = U_{\epsilon_0}$.

c) Use a) and b) to prove (1).

3. Consider the vector fields X, Y, Z on S^2 given by

$$X(p) = \xi \times p, \quad Y(p) = \eta \times p, \quad Z(p) = (\xi \times p) \times p,$$

for $\xi, \eta \in S^2$ and $\xi \neq \eta$. Calculate the Lie brackets $[X, Y]$, $[X, Z]$ and $[Y, Z]$.

Hint: Take $\xi = e_3$ to ease the calculations.

4. Assume X, Y, Z are complete vector fields on M.

a) Let φ, ψ be diffeomorphisms on M. Prove that $\varphi^*\psi^*X = (\psi \circ \varphi)^*X$.

b) Let φ be a diffeomorphism on M and let ψ^t be the flow of Y for $t \in \mathbb{R}$. Prove that $\varphi^{-1} \circ \psi^t \circ \varphi$ is the flow of $\varphi^* Y$ for $t \in \mathbb{R}$.

c) Let φ be a diffeomorphism on M. Prove that $\varphi^*[X, Y] = [\varphi^* X, \varphi^* Y]$.

d) Prove the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Hint: For c), use the formula $[X, Y] = \frac{d}{dt}\Big|_{t=0} (\psi^t)^* X$ where ψ^t if the flow of Y together with a) and b). For d), look at c) with φ replaced by φ^t, where φ^t is the flow of Z and differentiate with respect to $t \in \mathbb{R}$. Recall that the Lie bracket is a bi-linear map.

5. A car is parking on the side of the street. Its present position is given by its position $(x_1(t), x_2(t)) \in \mathbb{R}^2$ and the direction $\theta(t)$. (Here $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the angle of the axis of the car with the x_1 coordinate axis.) We assume that the velocity of the car always points in the direction of the car's axis. We consider the vector fields $X(x_1, x_2, \theta) := (\cos(\theta), \sin(\theta), 1)$, $Y(x_1, x_2, \theta) := (\cos(\theta), \sin(\theta), -1)$ on the phase space $P := \mathbb{R}^2 \times S^1$.

a) Show that X, Y are complete and calculate their flows $\varphi^t, \psi^t : P \to P$ for $t \in \mathbb{R}$. Also consider the curve $\chi : \mathbb{R} \to \mathrm{Diff}(P)$ given by

$$\chi(t) := \varphi^t \circ \psi^t \circ \varphi^{-t} \circ \psi^{-t}.$$

Calculate $\dot{\chi}(t)$ for $t \in \mathbb{R}$ and $1/2\, \ddot{\chi}(0)$.

b) Calculate the vector field $[X, Y]$ and the flow of $[X, Y]$.

6. Let $\mathbb{H} \cong \mathbb{R}^4$ be the quaternions, $Sp(1) = S^3$ be the group of unit quaternions and $\mathrm{Im}\,\mathbb{H} := \{x_1 i + x_2 j + x_3 k : (x_1, x_2, x_3) \in \mathbb{R}^3\} \cong \mathbb{R}^3$ be the imaginary quaternions. Recall that $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$ and $x^{-1} = \frac{\bar{x}}{|x|^2}$.

a) Let $\Phi : Sp(1) \to SO(3)$ be the map $x \mapsto \Phi(x)$ with

$$\Phi(x) : \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3 \to \mathrm{Im}\,\mathbb{H} \cong \mathbb{R}^3 \text{ is given by } \Phi(x)\xi = x\xi\bar{x}.$$

Write $\Phi(x)$ as a matrix and prove that this is a group homomorphism and a smooth double cover. Compute $d\Phi(1) : \mathfrak{sp}(1) = \mathrm{Im}\,\mathbb{H} \mapsto \mathfrak{so}(3)$ and conclude that this is a Lie algebra isomorphism. What is the Lie bracket on $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ you get?

b) Let $\Psi : Sp(1) \to SU(2)$ be the map $x \mapsto \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$. Prove that this is a group isomorphism and compute $d\Psi(1) : \mathfrak{sp}(1) \to \mathfrak{su}(2)$.

Thus we proved one of the so called accidental isomorphism

$$\mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3).$$