Exercise Sheet 4

Please hand in your solutions by October 16, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let $f : N \to M$ be a smooth map between two manifolds. Show that the following subsets of N are open:

$$U := \{p \in N \ : \ df(p) : T_pN \to T_{f(p)}M \text{ is surjective}\},$$
$$V := \{p \in N \ : \ df(p) : T_pN \to T_{f(p)}M \text{ is injective}\}.$$

Hint: Prove first that the sets $\{A \in \mathbb{R}^{m\times n} \ : \ \mathrm{rk}(A) = m\}$ and $\{A \in \mathbb{R}^{m\times n} \ : \ \ker(A) = 0\}$ are open subsets of $\mathbb{R}^{m\times n}$. Use this to prove the result in local coordinates.

2. Let M be a manifold and denote by $\mathcal{F}(M) = C^\infty(M, \mathbb{R})$ the space of smooth functions on M. Recall that every vector field $X \in \mathrm{Vect}(M)$ defines a derivation by the following formula

$$\mathcal{L}_X : \mathcal{F}(M) \to \mathcal{F}(M), \qquad (\mathcal{L}_X f)(p) := df(p)X(p).$$

Verify the identity

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_Y\mathcal{L}_X - \mathcal{L}_X\mathcal{L}_Y = -[\mathcal{L}_X, \mathcal{L}_Y],$$

(where the last equation holds by definition).

3. The aim of this exercise is to prove that for every derivation $\delta : \mathcal{F}(M) \to \mathcal{F}(M)$, there exists a unique smooth vector field X such that $\delta = \mathcal{L}_X$.

a) Let $f : B_\epsilon(a) \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Prove the formula

$$f(x) = f(a) + \sum_{i=1}^{n}(x_i - a_i)g_i(x) \quad \text{with} \quad g_i(x) = \int_0^1 (\partial_{x_i} f)(a + t(x-a))\, dt.$$

b) Let δ be a derivation and $f \in \mathcal{F}(M)$. Prove the following

(i) **(Constants)** If $f \equiv c$ is constant, then $\delta(f) = 0$.

(ii) **(Localisation)** If $p \notin \mathrm{supp}(f)$, then $\delta(f)(p) = 0$.

(iii) **(Vanishing)** If $df(p) = 0$, then $\delta(f)(p) = 0$.

c) Use the properties of part b) to prove the statement.

Hint: **(Localization)** implies $\delta(f)(p) = \delta(g)(p)$ if f and g agree in a neighborhood of p. Hence we may assume for the proof of (**Vanishing**), that f is supported in a chart centred at p and then use the formula from a).

4. Let M be a compact manifold. An automorphism of $\mathcal{F}(M)$ is a bijective $\mathbb{R}$-linear map $\Phi : \mathcal{F}(M) \to \mathcal{F}(M)$ satisfying

$$\Phi(fg) = \Phi(f)\Phi(g), \qquad \Phi(1) = 1.$$

The goal of this exercise is to show, that for every automorphism Φ there exists a diffeomorphism $\varphi : M \to M$ such that

$$\Phi(f) = f \circ \varphi =: \varphi^* f.$$

a) Show that every maximal ideal J of the ring $\mathcal{F}(M)$ has the shape

$$J_p = \{f \in \mathcal{F}(M) \,|\, f(p) = 0\}$$

for some $p \in M$.

b) Show that there exists a bijective map $\varphi : M \to M$ such that

$$\Phi^{-1}(J_p) = J_{\varphi(p)}, \qquad \Phi(J_p) = J_{\varphi^{-1}(p)},$$

and deduce from this that $\Phi(f) = f \circ \varphi$.

c) Suppose $\varphi : M \to M$ is a function such that $f \circ \varphi$ is smooth function for every $f \in \mathcal{F}(M)$. Show that φ is smooth.

Hint: For a): It is a result from abstract algebra that every ideal of a ring is contained in a maximal ideal. Now argue by contradiction: Suppose $J \subset \mathcal{F}(M)$ is an ideal and for every point $p \in M$ there exists a function $f_p \in J$ with $f(p) \neq 0$. By compactness there exists a finite linear combination $f = \sum_{i=1}^N f_{p_i}\rho_i$ which for suitable cutoff functions ρ_i has no zeros. Deduce from this the contradiction $J = \mathcal{F}(M)$.

5. Let $V \subset \mathbb{R}^\ell$ be a linear subspace.

a) Show that there exists a unique matrix $\Pi \in \mathbb{R}^{\ell\times\ell}$ satisfying

$$\Pi = \Pi^2 = \Pi^\top, \qquad \mathrm{Im}(\Pi) = V.$$

b) Let Π be as in a). Show that $\mathbb{R}^\ell = \ker(\Pi) \oplus \mathrm{Im}(\Pi)$ and $\ker(\Pi) \perp \mathrm{Im}(\Pi)$ are orthogonal subspaces.

c) Suppose $D \in \mathbb{R}^{\ell\times n}$ is injective and $V = \mathrm{Im}(D)$. Then $D^\top D$ is invertible and the matrix Π from a) is given by

$$\Pi = D(D^\top D)^{-1} D^\top.$$

6. Consider the following subset of $\mathbb{C}^2$:

$$E_{\text{Möb}} = \left\{ (e^{\mathbf{i}t}, \zeta) \in S^1 \times \mathbb{C} \,:\, \zeta \in \mathbb{R}e^{\mathbf{i}t/2} \right\}.$$

a) Show that $E_{\text{Möb}}$ is a rank 1 vector bundle over S^1.

b) Does there exists a section $\sigma : S^1 \to E_{\text{Möb}}$ with $\sigma(z) \neq 0$ fÃ$\frac{1}{4}$r all $z \in S^1$?

Hint: In a) Theorem 2.6.8 might be useful.

7. For $i = 1, 2$ let $E_i \subset M \times \mathbb{R}^{\ell_i}$ be vector bundles with natural projections $\pi_i : E_i \to M$. A smooth map $\Phi : E_1 \to E_2$ is called a vector bundle isomorphism if $\pi_2 \circ \Phi = \pi_1$ and for every $p \in M$ the maps $\Phi_p : (E_1)_p \to (E_2)_p$ defined by

$$(p, \Phi_p v) = \Phi(p, v), \qquad \forall v \in (E_1)_p.$$

are linear and bijective. The vector bundles E_1 and E_2 are called isomorphic, if there exists a vector bundle isomorphism $\Phi : E_1 \to E_2$.

Consider the vector bundles $E_{\text{Möb}}$, TS^1, $S^1 \times \mathbb{R}$ over S^1. Which of these bundles are isomorphic?

8. Let $E \subset M \times \mathbb{R}^\ell$ be vector bundle over M of rank r.

a) Let $f : N \to M$ be a smooth function. Show that

$$f^*E := \{(p, v) \in N \times \mathbb{R}^\ell \,:\, v \in E_{f(p)}\}$$

is a vector bundle over N of rank r.

b) Show that

$$E^\perp := \{(p, v) \in M \times \mathbb{R}^\ell \,:\, v \in E_p^\perp\}$$

is a smooth vector bundle over M. What is its rank?

c) For $i = 1, 2$ let $E_i \subset M \times R^{\ell_i}$ be vector bundles over M of rank r_i. Show that

$$E_1 \oplus E_2 := \{(p, v_1, v_2) \,:\, p \in M,\ v_1 \in (E_1)_p,\ v_2 \in (E_2)_p\} \subset M \times \mathbb{R}^{\ell_1 + \ell_2}$$

is a vector bundle over M of rank $r_1 + r_2$.

Hint: Use Theorem 2.6.8. to verify that these spaces are again vector bundles.