Exercise Sheet 5

Please hand in your solutions by October 23, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Prove that every one-dimensional vector subbundle $E \subset TM$ is integrable.

2. Consider the manifold $M = \mathbb{R}^3$. Prove that the subbundle $E \subset TM = \mathbb{R}^3 \times \mathbb{R}^3$ with fiber $E_p = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \zeta - y\xi = 0\}$ over $p = (x, y, z) \in \mathbb{R}^3$ is not integrable and that any two points in $\mathbb{R}^3$ can be connected by a path tangent to E.

Hint: Try to find two vector fields X, Y which span E at every point. Follow the flow lines of X and Y to find the path.

3. Let $M \subset \mathbb{R}^{m+1}$ be a submanifold of codimension one with a normal vector field $\nu : M \to \mathbb{R}^{n+1}$ i.e. $\nu(p) \perp T_pM$ for every $p \in M$ and $|\nu(p)| = 1$.

a) Give a formula for the second fundamental form in terms of ν.

b) Use this formula to derive the second fundamental form of $S^n \subset \mathbb{R}^{n+1}$.

4. Let $p \in M \subset \mathbb{R}^k$ and $v \in T_pM$.

a) Prove that $(d\Pi(p)v)\xi \in T_pM$ for all $v \in T_pM$ and $\xi \in T_pM^\perp$.

b) Prove that the adjoint of $h_p(v, \cdot)$ is given by

$$h_p(v, \cdot)^* : T_pM^\perp \to T_pM, \qquad \xi \mapsto (d\Pi(p)v)\xi.$$

Hint: Consider the following setup for a): Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $\eta : (-\epsilon, \epsilon) \to TM^\perp$ be a smooth section with $\eta(t) \in T_{\gamma(t)}M$ and let $\xi \in T_pM^\perp$. Proceed by differentiating the equation $0 = \langle \Pi(\gamma(t))\xi, \eta(t)\rangle$.

5. Let $M \subset \mathbb{R}^n$ be an m-manifold. Fix a point $p \in M$ and a unit tangent vector $v \in T_pM$ so that $|v| = 1$ and define

$$L := \{p + tv + w \,|\, t \in \mathbb{R}, w \perp T_pM\}.$$

Let $\gamma : (-\epsilon, \epsilon) \to M \cap L$ be a smooth curve such that $\gamma(0) = p, \dot\gamma(0) = v$, and $|\dot\gamma(t)| = 1$ for all t. Prove that

$$\ddot\gamma(0) = h_p(v, v).$$

Draw a picture of M and L in the case $n = 3$ and $m = 2$.

Hint: Write $\gamma(t) = p + \alpha(t)v + w(t)$ and show that $\ddot\gamma(0) = \ddot w(0) = h_p(v, v)$.

6. Choose a splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and write the elements of $\mathbb{R}^n$ as tuples $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$. Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold such that $p = 0 \in M$ and

$$T_0M = \mathbb{R}^m \times \{0\}, \quad T_0M^\perp = \{0\} \times \mathbb{R}^{n-m}.$$

By the implicit function theorem, there are open neighbourhoods $\Omega \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-m}$ of zero and a smooth map $f : \Omega \to V$ such that

$$M \cap (\Omega \times V) = \mathrm{graph}(f) = \{(x, f(x)) : x \in \Omega\}.$$

Thus $f(0) = 0$ and $df(0) = 0$. Prove that the second fundamental form $h_p : T_pM \times T_pM \to T_pM^\perp$ is given by the second derivatives of f, i.e.

$$h_p(v, w) = \left(0, \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right)$$

for all $v, w \in T_pM = \mathbb{R}^m \times \{0\}$.

Hint: Consider the vector fields $X(x, y) = (x, df(x)v)$ and $Y(x, y) = (x, df(x)w)$ and use that $h_p(v, w)$ agrees with the vertical component of $dX(p)Y(p)$.