# Exercise Sheet 6

Please hand in your solutions by October 30, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

**1.** Recall that $\mathcal{O}(M)$ is the orthonormal frame bundle which as a set is equal

$$\mathcal{O}(M) := \{(p,e) \in \mathbb{R}^n \times \mathbb{R}^{n\times m} : p \in M, \operatorname{im} e = T_pM, e^\top e = \mathbb{1}_{m\times m}\}.$$

**a)** Prove that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$.

**b)** Let $(p,e) \in \mathcal{O}(M)$. What is the tangent space $T_{(p,e)}\mathcal{O}(M)$ and what is its dimension?

**c)** Prove that the map $\pi : \mathcal{O}(M) \to M$ is a submersion.

**d)** Prove that the action of $GL(m)$ on $\mathcal{F}(M)$ restricts to an action of the orthogonal group $O(m)$ on $\mathcal{O}(M)$ whose orbits are the fibers

$$\mathcal{O}(M)_p := \{e \in \mathbb{R}^{n\times m} : (p,e) \in \mathcal{O}(M)\} = \{e \in \mathcal{L}_{iso}(\mathbb{R}^m, T_pM) : e^T e = \mathbb{1}\}$$

**Hint:** In a), try to see $\mathcal{O}(M) \subset \mathcal{F}(M)$ as the level set of a regular value.

**2.** Prove that $H_{(p,e)} \subset T_{(p,e)}\mathcal{O}(M)$ and that

$$T_{(p,e)}\mathcal{O}(M) = H_{(p,e)} \oplus V'_{(p,e)}; \quad V'_{(p,e)} = V_{(p,e)} \cap T_{(p,e)}\mathcal{O}(M),$$

for every $(p,e) \in \mathcal{O}(M)$.

**3.** **[Parallel transport on $S^2$]** Let $\alpha \in [0, 2\pi]$. Consider the following curves on $S^2$

$$\begin{aligned}
\gamma_1(t) &:= (\sin(t), 0, \cos(t)), \quad t \in [0, \pi/2],\\
\gamma_2(t) &:= (\cos(t), \sin(t), 0), \quad t \in [0, \alpha],\\
\gamma_3(t) &:= (\cos(t)\cos(\alpha), \cos(t)\sin(\alpha), \sin(t)), \quad t \in [0, \pi/2].
\end{aligned}$$

**a)** Draw the images of $\gamma_i$, $i = 1, 2, 3$.

**b)** Let $v \in T_{(0,0,1)}S^2$. What is the parallel transport of $v$ along $\gamma_1$, i.e. $\Phi_{\gamma_1}(t, 0)v$?

**c)** Let $\gamma$ be the piecewise smooth curve, which we obtain by going along $\gamma_1$, then $\gamma_2$ and at the end $\gamma_3$. What is the parallel transport along $\gamma$, i.e. $\Phi_\gamma(\pi + \alpha, 0)v$, for $v \in T_{(0,0,1)}S^2$.

**Hint:** Think of parallel transport as a mighty soldier carrying his lance in front of him as he walks along the curves without ever turning at the corners.

**4.** The first fundamental from on a manifold $M^m \subset \mathbb{R}^n$ is defined by

$$g : TM \oplus TM \to \mathbb{R}, \ (p, v \oplus w) \mapsto g_p(v, w) := \langle v, w \rangle.$$

**a)** Let $\psi : \Omega \subset \mathbb{R}^m \to U \subset M$ be a local parametrisation. The functions $g_{ij} : \Omega \to \mathbb{R}, \ i, j = 1, \ldots, m$ are defined by

$$g_{ij}(x) := \sum_{\nu=1}^{n} \frac{\partial \psi^\nu}{\partial x^i}(x) \frac{\partial \psi^\nu}{\partial x^j}(x). \tag{1}$$

How are the matrix $(g_{ij})_{i,j=1,\ldots,n}$ and the first fundamental form $g$ related?

**b)** Let $\varphi : S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection. Find a formula for $\varphi$ and its inverse $\psi : \mathbb{R}^n \to S^n \setminus \{N\}$. Find the metric $g_{ij}, \ i, j = 1, \ldots, n$ in these local coordinates.

**5.** Let $M^m \subset \mathbb{R}^n$ be a manifold. Let $\psi : \Omega \to U \subset M$ be a local parametrisation. Let $c : I \to \Omega$ be a curve and define $\gamma := \psi \circ c$. Any vector field $X : I \to \mathbb{R}^n$ along $\gamma$ can be written as $X(t) = \sum_{k=1}^m \xi^k(t) \frac{\partial \psi}{\partial x^k}(c(t))$.

Define the **Christoffel symbols** $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ for $i, j, k = 1, \ldots, m$ by

$$\Pi(\psi(x)) \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x) =: \sum_{k=1}^{m} \Gamma_{ij}^k(x) \frac{\partial \psi}{\partial x^k}(x).$$

**a)** The vector field $\nabla X : I \to \mathbb{R}^n$ is defined by $\nabla X(t) = \Pi(\gamma(t))\dot{X}(t)$ for all $t \in I$. This can be expressed as $\nabla X(t) = \sum_{k=1}^m \eta^k(t) \frac{\partial \psi}{\partial x^k}(c(t))$. What is the relationship between $\eta^k$, $\xi^k$ and $\Gamma_{ij}^k$?

**b)** Define the matrix $(g^{ij}(x))$ to be the inverse of $(g_{ij}(x))$ for $x \in \Omega$. Define $\Gamma_{kij} : \Omega \to \mathbb{R}$ for $i, j, k \in 1, \ldots, m$ by $\Gamma_{ij}^\ell = \sum_{k=1}^m g^{\ell k} \Gamma_{kij}$. Prove that for $i, j, k \in 1, \ldots, m$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}.$$

**c)** Prove that

$$\Gamma_{kij} = \Gamma_{kji}, \quad \text{and} \quad \Gamma_{kij} = \frac{1}{2}\left( \frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

This last equation says that the Christoffel symbols are 'intrinsic'.

**d)** Use a) to find another proof of short time existence for parallel transport along a curve $\gamma$ as in Theorem 1 (3.1.16. in the lecture notes). I.e. prove that for $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$, there is $\epsilon > 0$ such that $I_0 := (t_0 - \epsilon, t_0 + \epsilon) \subset I$ and such that there is a unique $X \in \mathrm{Vect}(\gamma|_{I_0})$ such that $\nabla X = 0$ and $X(t_0) = v_0$.

**Hint:** For b), use $\frac{d}{dt}\langle X, Y \rangle = \langle \nabla X, Y \rangle + \langle X, \nabla Y \rangle$ for suitable $X, Y \in \mathrm{Vect}(\gamma)$ as a starting point. In c) use the symmetry (1. equation) and b) to get to the 2. equation. Think of how you would integrate $e^t \cos(t)$.

**6.** Consider the case $m = 2$. Let $\Omega \subset \mathbb{R}^2$ be an open set and $\lambda : \Omega \to (0, +\infty)$ be a smooth function. Suppose that the metric $g : \Omega \to \mathbb{R}^{2\times 2}$ is given by

$$g(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}.$$

**a)** Find the Christoffel symbols $\Gamma_{ij}^k$ by using the formula in Exercise 5 c).

**b)** For $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} := \mathbb{H}$ and $\lambda(x_1, x_2) = \frac{1}{x_2^2}$, calculate $\Gamma_{ij}^k$.

$(\mathbb{H}, g)$ is called Poincaré half-plane model for two dimensional hyperbolic geometry.