Exercise Sheet 7

Please hand in your solutions by October 6, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. [Development of the cylinder Z] Consider the cylinder $Z := \mathbb{R} \times S^1$ and let

$$\gamma : [0,1] \to Z, \qquad \gamma(t) = (h(t), e^{i\theta(t)})$$

be a smooth curve. Let $p_0 \in \mathbb{R}^2$ and $\Psi_0 : T_{\gamma(0)}Z \to \mathbb{R}^2$ be an orthogonal isomorphism.

a) Find the development (Ψ, γ, γ') of Z along $\mathbb{R}^2$ with $\gamma'(0) = p_0$, $\Psi(0) = \Psi_0$.

b) Give an example of a closed curve $\gamma : [0,1] \to Z$, $\gamma(0) = \gamma(1)$, such that γ' is not closed.

c) From this example give a geometric interpretation of the term development.

Hint: The path of orthogonal isometries $\Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t) : T_{\gamma(t)}Z \to \mathbb{R}^2$ does not depend on $\gamma'(t)$ (as parallel transport in $\mathbb{R}^2$ is trivial). Hence, one can first solve for $\Phi(t)$ and then recover $\gamma(t)$ from the equation $\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$.

2. [Development of S^2 along $\mathbb{R}^2$] Let $\gamma : [0, \pi + \alpha] \to S^2$ be the curve from the exercise 3 c) of exercise sheet 6. Let $\Psi_0 : T_{(0,0,1)}S^2 \to T_0\mathbb{R}^2$ be the orthogonal isomorphism given by

$$\Psi_0\,(1,0,0) := (1,0) \qquad \Psi_0\,(0,1,0) := (0,1)\,.$$

a) Find the development (Ψ, γ', γ) of S^2 along $\mathbb{R}^2$ with $\gamma'(0) = p'_0$, $\Psi(0) = \Psi_0$.

b) For which $\alpha \in [0, 2\pi)$ is the curve γ' in *a*) closed?

Hint: Proceed as in Exercise 1 to calculate the developement. The curve γ' and Ψ will be just piecewise smooth and we require the condition $\Psi(t)\dot\gamma(t) = \dot\gamma'(t)$ to hold only for $t \neq \pi/2,\ \pi/2 + \alpha$.

3. [Surface of revolution] Let $I \subset \mathbb{R}$ be an interval and let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ be a smooth curve. Define the map $\psi : \mathbb{R}/2\pi\mathbb{Z} \times I \to \mathbb{R}^3$ by

$$\psi(s,t) := (\gamma_1(t)\cos(s), \gamma_1(t)\sin(s), \gamma_2(t)).$$

a) Draw the image of ψ, $\operatorname{im}(\psi) = S$ and give some examples of the surfaces of revolution.

b) When is ψ an immersion? When is it an embedding?

c) In the case when ψ is an embedding, calculate the metric g_{ij} from exercise 4 of exercise sheet 6.

d) In the case when ψ is an embedding, calculate the Christoffel symbols Γ_{ij}^k from exercise 5 of exercise sheet 6.

4. For $p, q \in M$ define

$$\Omega_{p,q} := \Omega_{p,q}(M) := \{\gamma \in C^\infty([0,1], M) : \ \gamma(0) = p, \ \gamma(1) = q\}.$$

A map $\mathbb{R} \to \Omega_{p,q}$, $s \mapsto \gamma_s$ is called smooth path of curves, if the map joint map $\mathbb{R} \times [0,1] \to M$, $(s,t) \mapsto \gamma_s(t)$ is smooth as a function of s and t.

a) Let $\gamma : \mathbb{R} \to \Omega_{p,q}$ be a smooth path of curves. Prove that $X := \frac{d}{ds}\Big|_{s=0} \gamma_s \in \operatorname{Vect}(\gamma_0)$ with $X(0) = X(1) = 0$.

b) Conversely, let $\gamma_0 \in \Omega_{p,q}$ and $X \in \operatorname{Vect}(\gamma_0)$ with $X(0) = X(1) = 0$ be given. Prove that there exists a smooth path of curves $\gamma : \mathbb{R} \to \Omega_{p,q}$, passing through γ_0 at $s = 0$, such that $X := \frac{d}{ds}\Big|_{s=0} \gamma_s$.

This proves the identification

$$T_\gamma \Omega_{p,q} = \{X \in \operatorname{Vect}(\gamma) : \ X(0) = 0, \ X(1) = 0\}.$$

5. Let $U \subset \mathbb{R}^m$ be an open set and $g = (g_{ij}) : U \to \mathbb{R}^{m \times m}$ be a smooth map with values in the space of positive definite symmetric matrices. Let $p, q \in U$ with $p \neq q$.

a) For a smooth function $L : U \times \mathbb{R}^m \to \mathbb{R}$ consider the functional

$$E : \Omega_{p,q}(U) \to \mathbb{R}, \qquad E(\gamma) := \int_0^1 L(\gamma(t), \dot\gamma(t))\, dt$$

on the space of paths $\Omega_{p,q}(U)$ in U. Seeing E as a function on $\Omega_{p,q}$, prove that critical points of E are exactly those paths that fulfill the Euler-Lagrange equations of this variational problem which have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \xi^k}(\gamma(t), \dot\gamma(t)) = \frac{\partial L}{\partial x_k}(\gamma(t), \dot\gamma(t)), \quad k = 1, \cdots m, \quad t \in [0,1]. \tag{1}$$

Here we denote $L = L(x, \xi)$ with $(x, \xi) \in U \times \mathbb{R}^m$.

b) Specialise to the case

$$L(x,\xi) := \frac{1}{2}\sum_{i,j=1}^{m} \xi^i g_{ij}(x)\xi^j.$$

In this case, we call E the energy functional. Prove that the Euler-Lagrange equations (**??**) are equivalent to the geodesic equations

$$\ddot{\gamma}^k(t) + \sum_{i,j=1}^{m} \Gamma_{ij}^k(\gamma(t))\,\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0, \quad k = 1,\cdots,m, \quad t \in [0,1]$$

where Γ_{ij}^k are Christoffel symbols.

Hint: For a), we consider $\Omega_{p,q}$ formally as a manifold and calculate for a tangent vector $X := \frac{d}{ds}\big|_{s=0}\gamma_s \in \mathrm{Vect}(\gamma)$ the derivative by $dE(\gamma)X = \frac{d}{ds}\big|_{s=0} E(\gamma_s)$.

6. Let Arccos : $[-1,1] \to [0,\pi]$ denote the inverse of the function $\cos|_{[0,\pi]}$. Consider the function on S^2 given by

$$d(p,q) = \mathrm{Arccos}(\langle p,q\rangle), \qquad \text{for } p,q \in S^2.$$

a) Prove d is the intrinsic distance function on S^2, i.e. $d(p,q) = \inf L(\gamma)$ where the infimum is taken over all curves connecting p to q.

b) Prove that d induces the same topology as the subset topology on $S^2 \subset \mathbb{R}^3$.

Hint: In a), use spherical coordinates and take adventage of the symmetrie of the sphere. For b), you could prove equivalence to another distance function on S^2 which induces the subset topology.