Exercise Sheet 8

Please hand in your solutions by November 13, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let I be an interval and $\gamma \in C^\infty(I, S^n)$. Show that the following are equivalent:

(i) γ is a geodesic.

(ii) $|\dot\gamma|$ is constant and $\mathrm{im}(\gamma) \subset S^n$ is a big circle, i.e. there exists a $2-$plane $P \subset \mathbb{R}^{n+1}$ through the origin such that $\mathrm{im}(\gamma) \subset P \cap S^n$.

Hint: To calculate the exponential map, use the second fundamental form $h_p(v, w) = -p\langle v, w\rangle$ calculated in exercise 3 of sheet number 5.

2. Let ψ be a parametrisation of the surface of revolution S, as in exercise 3 of sheet number 7. The image of ψ of the curves $s =$ const. and $t =$ const. are called *meridians* and *parallels*. Suppose that the curve γ in the definition of ψ is parametrised by arc length i.e. $|\dot\gamma| = 1$. Prove and discuss the following:

a) The meridians, i.e. the curves $\psi_s(t) : I \to \mathbb{R}^3$, $\psi_s(t) = \psi(s, t)$ are geodesics on S.

b) When is a parallel $\psi^t : \mathbb{R} \to \mathbb{R}^3$, $\psi^t(s) := \psi(s, t)$ a geodesic?

c) Describe the geodesics on the standard cylinder in $\mathbb{R}^3$,

$$Z := \{(x, y, z) \in \mathbb{R}^3 \,|x^2 + y^2 = 1\}.$$

3. Let $p_0 \in M^m$. We know from the lectures that there exists $\epsilon > 0$ and a neighbourhood $U_\epsilon \subset M$ of the point p_0 such that

$$\exp_{p_0} : \{v \in T_{p_0}M : \ |v| < \epsilon\} \to U_\epsilon$$

is a diffeomorphism. Let $\{e_i\}_{i=1,\cdots m}$ be an orthonormal basis of $T_{p_0}M$. We define a diffeomorphism $\psi : B_\epsilon(0) \subset \mathbb{R}^m \to U_\epsilon$ by

$$\psi(x) := \exp_{p_0}\Big(\sum_{i=1}^m x^i e_i\Big).$$

a) Show that $g_{ij}(0) = \delta_{ij}$, for $i, j = 1, \cdots m$.

b) Show that $\Gamma^k_{ij}(0) = 0$ for $i, j, k = 1, \cdots m$.

These coordinates are called *normal coordinates centered at p_0*.

4. Determine the radius of injectivity at all points of the following manifolds.

a) $T^2 := S^1 \times S^1$. **b)** $\mathbb{R}^2 \setminus \{0\}$. **c)** S^n.

Hint: For c), use the exponential map which you should already have calculated in exercise 1.

5. Find an example of a manifold M such that any two points $p, q \in M$ can be connected by a geodesic. Find an example of a manifold that does not have this property.

6. Recall from Exercise Sheet 1, that $O(n)$ is a manifold with tangent spaces

$$T_gO(n) = \{\xi \in \mathbb{R}^{n\times n} : g^\top \xi + \xi^\top g = 0\} \quad \text{for } g \in O(n).$$

a) The euclidean inner product on $\mathbb{R}^{n\times n}$ is defined by

$$\langle \xi, \eta \rangle := \mathrm{tr}(\xi^\top \eta).$$

Verify the following formulae.

(i) $\Pi(g)\xi = \frac{1}{2}(\xi - g\xi^\top g)$ for all $g \in O(n)$ and $\xi \in \mathbb{R}^{n\times n}$.,

(ii) $h_g(\xi, \eta) = -\frac{1}{2}g(\xi^\top \eta + \eta^\top \xi)$ for all $g \in O(n)$ and $\xi, \eta \in T_gO(n)$.

b) Recall the matrix exponential given by $\exp(\xi) = \sum_{k=0}^\infty \frac{\xi^k}{k!}$. Prove that the geodesic γ starting at $g \in O(n)$ with initial velocity $\dot\gamma(0) = \xi \in T_gO(n)$ is given by

$$\gamma(t) = g\exp(tg^\top \xi) \quad \text{for } t \in \mathbb{R}. \tag{1}$$

In other words, the (geodesic) exponential map at the identity $\exp_{\mathbb{1}}$ is the (matrix) exponential map.

c) Let $G \subset O(n)$ be a compact Lie subgroup. Prove that (1) still holds for all $g \in G$ and $\xi \in T_gG$.

(A theorem from Lie theory says that closed subgroups of a Lie group are also submanifolds.)

d) Determine

$$r_0 := \inf\{||\xi|| \,:\, 0 \neq \xi \in \exp_{\mathbb{1}}^{-1}(\mathbb{1})\}.$$

(It is a convincing and true fact that the injectivity radius of $O(n)$ is given by $\frac{r_0}{2}$. However, the proof is quite sophisticated.)

Hint: In b) use the Gauss-Weingarten formula to deduce that $\dot\gamma(t)^\top \dot\gamma(t)$ is constant and then verify that $g\exp(tg^\top\xi)$ satisfies the same ODE as $\gamma(t)$. For c) recall that in a small neighbourhood of a point, geodesics are characterised as being length minimising curves. For d), use a base change to get a skew-symmetric matrix into its normal form with diagonal blocks $\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.