Exercise Sheet 9

Please hand in your solutions by November 20, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let (M, d) be a metric space. Prove that the following are equivalent:

(i) Every closed and bounded subset $A \subset M$ is compact.

(ii) Every bounded sequence in M has a convergent subsequence.

2. Let $M \subset \mathbb{R}^k$ be a manifold and d the intrinsic distance function on M.

a) Assume M is closed as a subset of $\mathbb{R}^k$ and prove that (M, d) is complete.

b) Give an example of a manifold $M \subset \mathbb{R}^k$ which is complete but not closed.

c) Prove that if $M \subset \mathbb{R}^k$ is compact then M is geodesically complete.

3. Let $M \subset \mathbb{R}^k$ be a manifold, $I \subset \mathbb{R}$ an open interval, and $\gamma : I \to M$ a geodesic. Fix a point $t_0 \in I$. Show that there exists $\epsilon > 0$ such that for any two real numbers $t_0 - \epsilon < s < t < t_0 + \epsilon$ the restriction of γ to the interval $[s, t]$ is length minimizing, i.e. $L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$.

How large can you choose ϵ in the case $M = S^2$?

Hint: Define $\epsilon := \frac{1}{2} \inf\{\mathrm{inj}(y) : d(y, \gamma(t_0)) \leq r\}$ and use Theorem 4.4.4 to show that this does the job when r is sufficiently small.

4. **a)** How large can you choose the constant $r(p) > 0$ for $p \in M$ such that $U_{r(p)} := \{q \in M \mid d(p, q) < r(p)\}$ is geodesically convex in the cases

(i) $M = T^2$, (ii) $M = \mathbb{R}^2 \setminus \{0\}$, (iii) $M = S^2$.

b) Find a geodesically convex set U in a manifold M and points $p, q \in U$ such that the unique geodesic $\gamma : [0, 1] \to U$ with $\gamma(0) = p$ and $\gamma(1) = q$ has length $L(\gamma) > d(p, q)$.

c) Find a set U in a manifold M such that any two points in U can be connected by a minimal geodesic in U, but U is not geodesically convex.

Hint: For b), take M to be the upper hemisphere $\{(x, y, z) \in \mathbb{R}^3 | \, x^2 + y^2 + z^2 = 1, z > 0\}$ together with the disc $\{(x, y, z) \in \mathbb{R}^3 | \, x^2 + y^2 \leq 1, z = 0\}$, but smooth the corners along the circle $x^2 + y^2 = 1$, $z = 0$. Take U to be a large metric ball for the intrinsic distance in the upper hemisphere.

5. Let $M \subset \mathbb{R}^3$ be a two dimensional manifold and suppose that M is invariant under the (orthogonal) reflection across some plane $E \subset \mathbb{R}^3$. Show that E intersects M in a union of geodesics. Conclude for example that the coordinate planes intersect the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ in geodesics.

Hint: If $E \cap M$ would not be the image of a geodesic, then there would be points $p, q \in M$ very close to one another joined by two distinct minimal geodesics.

6. Let $\gamma : I = [a, b] \to M$ be a smooth curve such that $\dot{\gamma}(t) \neq 0$ for all $t \in I$.

a) Define $\sigma : [a, b] \to [0, L(\gamma)]$ by

$$\sigma(t) := \int_a^t |\dot{\gamma}(s)| \, ds.$$

Prove that σ is a smooth diffeomorphism and that

$$\gamma' := \gamma \circ \sigma^{-1} : [0, L(\gamma)] \to M$$

is parametrized by arclength, i.e. $|\dot{\gamma}'(t)| = 1$ for all $t \in [0, T]$.

b) Prove that the derivative of the length functional at γ is given by

$$dL(\gamma)X = -\int_a^b \left\langle \dot{V}(t), X(t) \right\rangle dt, \qquad V(t) := \frac{\gamma(t)}{||\dot{\gamma}(t)||}$$

where $X \in \mathrm{Vect}(\gamma)$ with $X(a) = 0 = X(b)$.

c) Prove that γ is an extremal point of L if and only if the curve γ' from part a) is a geodesic.

Hint: Part a) provides a canonical parametrization for a geometric curve. Part b) depends on the assumption $\dot{\gamma}(t) \neq 0$ and the calculation is similar to the proof of Theorem 4.1.4. Part c) claims that a critical points of the length functional depend only on the geometrical curve and that the arclength parametrization yields a geodesic.