Exercise Sheet 10

Please hand in your solutions by November 27, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Let $M \subset \mathbb{R}^n$ be a m-dimensional manifold. Let $\psi : \Omega \to U \subset M$ be a local parametrisation. For $x \in \Omega$ define $b_i(x) = \frac{\partial \psi}{\partial x^i}(x)$ and $g_{ij}(x) := \langle b_i(x), b_j(x) \rangle$. Define $R^\ell_{ijk} : \Omega \to \mathbb{R}$ by

$$R(b_i, b_j)b_k =: \sum_{\ell=1}^m R^\ell_{ijk} b_\ell.$$

Show that

$$R^\ell_{ijk} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \sum_{\nu=1}^m (\Gamma^\ell_{i\nu}\Gamma^\nu_{jk} - \Gamma^\ell_{j\nu}\Gamma^\nu_{ik})$$

for all $\ell, i, j, k \in \{1, \cdots, m\}$ where Γ^k_{ij} are the usual Christoffel symbols, defined by $\nabla_{b_i} b_j = \sum_{k=1}^m \Gamma^k_{ij} b_k$.

b) Calculate R^ℓ_{ijk} for the stereographic projection on the sphere S^2.

Hint: For b), recall that we calculated in Exercise 4 of sheet 6, the metric in stereographic projection to be $g_{ij}(x) = \frac{4\delta_{ij}}{(1+|x|^2)^2}$. You can use the formulae for the Christoffel symbols from Exercise 6 of sheet 6. You only need to calculate R^1_{122} as the other components follow by symmetry.

2. Let $G \subset O(n)$ be a Lie subgroup of $O(n)$.

a) Every $\xi \in T_{\mathbb{1}}G = \mathrm{Lie}(G)$ determines a left-invariant vector field on G by the formula $X_\xi(g) := g\xi \in T_gG$. Show that

$$\nabla_{X_\xi} X_\eta = \frac{1}{2} X_{[\xi,\eta]}$$

where $[\xi, \eta] = \xi\eta - \eta\xi$ is the commutator Lie bracket.

b) Show that $[X_\xi, X_\eta] = X_{[\eta,\xi]}$.

c) Show that the Riemann curvature tensor of G is given by

$$R_g(g\xi, g\eta)g\zeta = -\frac{1}{4} g[[\xi, \eta], \zeta]$$

for $g \in G$ and $\xi, \eta, \zeta \in T_{\mathbb{1}}G$.

Hint: For a): Prove the formula first for $G = O(n)$ using Exercise 6 from Exercise Sheet 8. Why does it remain valid for any subgroup? For b): Use that ∇ is torsion free. For c): Extend the tangent vectors to left-invariant vector fields and use the relations from a) and b) to calculate the curvature tensor.

3. Let $X \in \mathrm{Vect}(M)$. Prove the following.

a) If the map $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ satisfies

$$\mathcal{L}_X \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle,$$

for all $Y, Z \in \mathrm{Vect}(M)$, then D_X is linear.

b) If the map $D : \mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto D_X$ satisfies

$$D_Y X - D_X Y = [X, Y],$$

for all $X, Y \in \mathrm{Vect}(M)$, then D is linear.

4. a) Prove that every isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map

$$\phi(p) = \Phi p + p_0,$$

where $p_0 \in \mathbb{R}^n$ and $\Phi \in O(n)$. Thus ϕ is a composition of rotation and translation.

b) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Suppose $M,\ M' \subset \mathbb{R}^n$ are manifolds such that $\phi(M) = M'$. Show that ϕ intertwines their second fundamental form:

$$(d\phi(p))^{-1} h'_{\phi(p)}(d\phi(p)v, d\phi(p)w) = h_p(v, w), \quad \forall v, w \in T_pM, \quad \forall p \in M.$$

Hint: For b): Assume that $\phi(p) = \Phi p + p_0$ and show first that the families of orthogonal projections on M and M' are related by $\Pi'(\phi(p)) = \Phi^t \Pi(p) \Phi$.

5. A complete vector field $X \in \mathrm{Vect}(M)$ is called a **Killing vector field** if its flow $\phi^t : M \to M$ is an isometry for every $t \in \mathbb{R}$.

a) Give examples of Killing vector fields.

b) Let X be a Killing vector field. Prove that

$$\langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0,$$

for $v, w \in T_pM,\ p \in M$.

c) Let X be a Killing field and $\gamma : I \to M$ a geodesic. Show that

$$\frac{d}{dt}\langle \dot\gamma, X(\gamma)\rangle = 0.$$

Hint: For a), you can try to find Killing fields for $M = \mathbb{R}^n$.

6. Prove the following.

a) Every compact connected 1− manifold is diffeomorphic to S^1.

b) Define the length of a compact connected 1−manifold.

c) Two compact connected 1−manifolds are isometric if and only if they have the same length.

Hint: For a): Let $\gamma : \mathbb{R} \to M$ be a geodesic with $|\dot{\gamma}| = 1$. First, show that γ is not injective. Second, show by contradiction for $t_0 < t_1$ with $\gamma(t_0) = \gamma(t_1)$ that $\dot{\gamma}(t_0) = \dot{\gamma}(t_1)$ (otherwise prove that $\gamma(t_0 + t) = \gamma(t_1 - t)$ for all $t \in \mathbb{R}$).