Exercise Sheet 11

Please hand in your solutions by December 4, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let $\Sigma \subset \mathbb{R}^3$ be a 2-manifold and $p_0 \in \Sigma$. Let $0 < \epsilon < \mathrm{inj}(p_0, \Sigma)$. Let $\gamma_\epsilon : [0,1] \to \Sigma$ be a parametrisation of the geodesic circle $C_\epsilon := \{p \in \Sigma : d(p, p_0) = \epsilon\}$ of radius $\epsilon > 0$ around p_0. Define $\ell_\epsilon := \mathrm{L}(\gamma_\epsilon)$ to be the length of γ_ϵ.

a) Prove that the following relation holds.

$$K(p_0) = -3 \left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \frac{\ell_\epsilon}{2\pi\epsilon}. \tag{1}$$

b) Use (1) for $\Sigma = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and $\Sigma = S^2 \subset \mathbb{R}^3$ to calculate their Gauss curvature.

Hint: For a), this is a long calculation in geodesic normal coordinates around p_0. In these coordinates, we have that $g_{ij}(0) = \delta_{ij}$, $\partial_k g_{ij}(0) = 0$ and $\partial_{11}g_{22}(0) = \partial_{22}g_{11}(0) = -2\partial_{12}g_{12}(0)$. All other second derivatives $\partial_{ij}g_{k\ell}(0)$ are zero. Also we can express $R^1_{122}(0)$ in terms of derivatives of $\partial_{11}g_{22}(0)$. In geodesic normal coordinates C_ϵ can be easily parametrised. After that, calculate until your fingers burn.

2. Let ψ be a parametrisation of the (embedded) surface of revolution S, as in exercise 3 of sheet number 7.

a) Show that the Gauss curvature of S is given by

$$K(\psi(s,t)) = \frac{R^1_{122}(s,t)}{g_{22}(s,t)} = \frac{-\dot\gamma_2(t)^2\ddot\gamma_1(t) + \dot\gamma_1(t)\dot\gamma_2(t)\ddot\gamma_2(t)}{\gamma_1(t)(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)^2}$$

b) Use part a) to calculate the Gauss curvature of

(i) The cylinder $Z = S^1 \times \mathbb{R} \subset \mathbb{R}^3$ (ii) The sphere $S^2 \subset \mathbb{R}^3$.

(iii) The torus $T^2 = \{(x,y,z) \in \mathbb{R}^3 : \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 = a^2\}$ for $0 < a < R$.

c) The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (1/\cosh(t), t - \tanh(t))$, is called *tractrix*. Its associated surface of revolution is *Beltrami's pseudosphere*:

$$\Sigma := \left\{ \left(\frac{\cos(s)}{\cosh(t)}, \frac{\sin(s)}{\cosh(t)}, t - \tanh(t) \right) \,\middle|\, t \in \mathbb{R},\ s \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Use part a) to show that $K(\psi(s,t)) = -1$ provided $t \neq 0$.

Hint: For part a) Use Theorem 5.3.7 on the Gaussian curvature and exercise 1 of exercise sheet 10 on the curvature tensor in local coordinates. We have calculated the Christoffel symbols already in exercise 3 of exercise sheet 7.

3. We call a manifold $M \subset \mathbb{R}^n$ **flat** if the Riemann curvature tensor R vanishes everywhere.

a) Prove that every $1-$ dimensional manifold is flat.

b) Let $M_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$ be flat manifolds. Prove that $M = M_1 \times M_2$ is flat.

c) Prove that a cylinder $Z = S^1 \times \mathbb{R} \subset \mathbb{R}^3$ and the tori $T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n$ are flat.

4. **a)** For a smooth curve $\gamma : \mathbb{R} \to M$ and a normal vector field $Y \in \mathrm{Vect}^\perp(\gamma)$ define the covariant derivate by

$$\nabla^\perp Y(t) = (\mathbb{1} - \Pi(\gamma(t)))\dot{Y}(t).$$

Then $\nabla^\perp Y \in \mathrm{Vect}^\perp(\gamma)$ is again a normal vector field along γ. Show that

$$\dot{Y}(t) = \nabla^\perp Y(t) - h_{\gamma(t)}(\dot{\gamma}(t))^* Y(t)$$

b) The curvature tensor of $TM^\perp$ is a collection of linear map $R_p^\perp : T_pM \times T_pM \to \mathcal{L}(T_pM^\perp, T_pM^\perp)$ defined by

$$R^\perp(\partial_s\gamma, \partial_t\gamma)Y = \nabla_s^\perp \nabla_t^\perp Y - \nabla_t^\perp \nabla_s^\perp Y$$

where $\gamma : \mathbb{R}^2 \to M$ denotes now a two parameter family and $Y \in \mathrm{Vect}^\perp(\gamma)$. Show that

$$R_p^\perp(u, v) = h_p(u)h_p(v)^* - h_p(v)h_p(u)^*$$

for all $p \in M$ and $v, w \in T_pM$.

Hint: We have shown in exercise 5 of exercise sheet 5 that $h_p(v)^* : T_pM^\perp \to T_pM$ is given by $h_p(v)^*\xi = (d\Pi(p)v)\xi$.

5. Prove that on a complete manifold, any Killing vector field is complete.

Hint: Start by proving that $|\dot{\gamma}|$ is constant along any flow line γ of X.

6. Let X, Y be Killing vector fields on a complete manifold M.

a) Prove that $X + \lambda Y$ is again a Killing vector field for all $\lambda \in \mathbb{R}$.

b) Prove that if there is $p \in M$ such that $X(p) = Y(p)$ and $\nabla_v X = \nabla_v Y$ for all $v \in T_pM$, then $X = Y$.

c) Give an upper bound on the dimension of the vector space of Killing fields.

d) Write down all Killing fields on $\mathbb{R}^3$.

Hint: For a), prove that a vector field verifying the relation $\langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0$, for all $p \in M$ and $v, w \in T_pM$ is a Killing field. For b), use a), to conclude that it is enough to prove that a Killing field with $X(p) = \nabla X(p) = 0$ has to be zero. Prove that $d\phi^t(p) = \mathbb{1}$ for all $t \in \mathbb{R}$ for the flow of X. Conclude by using the uniqueness of isometries. For c), bunching together all the $\nabla_v X$ will give you a linear map with a special property, so that you get a better bound than $m + m^2$.