Exercise Sheet 12

Please hand in your solutions by December 11, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Prove that S^n is simply connected for $n \geq 2$.

Hint: You may use that the image of a smooth curve $\gamma : [a, b] \to S^n$ has measure zero. This is a special case of Sard's theorem which says that for any smooth map $f : M \to N$ the set of regular values has full measure. This theorem will be a major result from the second semester.

2. (Developable hypersurfaces) Let $n = m + 1$ and let $E(t)$ be a one-parameter family of hyperplanes in $\mathbb{R}^n$. Then there is a smooth map $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$E(t) = u(t)^\perp, \qquad |u(t)| = 1$$

for every t. We assume that $\dot{u}(t) \neq 0$ for every t so that $u(t)$ and $\dot{u}(t)$ are linearly independent.

a) Show that

$$L(t) := u(t)^\perp \cap \dot{u}(t)^\perp = \lim_{s \to t} E(s) \cap E(t).$$

Thus $L(t)$ is a linear subspace of dimension $m - 1$.

b) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that

$$\langle \dot{\gamma}(t), u(t) \rangle = 0, \qquad \langle \dot{\gamma}(t), \dot{u}(t) \rangle \neq 0.$$

for all t. This means that $\dot{\gamma}(t) \in E(t)$ and $\dot{\gamma}(t) \notin L(t)$; thus $E(t)$ is spanned by $L(t)$ and $\dot{\gamma}(t)$. For $t \in \mathbb{R}$ and $\epsilon > 0$ define

$$L(t)_\epsilon = \{v \in L(t) \mid |v| < \epsilon\}.$$

Let $I \subset \mathbb{R}$ be a bounded open interval such that the restriction of γ to the closure of I is injective. Prove that, for $\epsilon > 0$ sufficiently small, the set

$$M_0 := \bigcup_{t \in I} (\gamma(t) + L(t)_\epsilon).$$

is a smooth manifold of dimension $m = n - 1$ and the spaces $L(t)_\epsilon$ are all disjoint for different t. A manifold which arises this way is called **developable**.

c) Show that the tangent spaces of M_0 are the original subspaces $E(t)$, i.e.

$$T_pM_0 = E(t) \quad \text{for} \quad p \in \gamma(t) + L(t)_\epsilon.$$

One therefore calls M_0 the *envelope* of the hyperplane $\gamma(t) + E(t)$.

d) Show that M_0 is flat.

e) If (Φ, γ, γ') is a development of M_0 along $\mathbb{R}^m$, show that the map $\phi : M_0 \to \mathbb{R}^m$, defined by

$$\phi(\gamma(t) + v) = \gamma'(t) + \Phi(t)v$$

for $v \in L(t)_\epsilon$ is an isometric immersion onto an open set $M_0' \subset \mathbb{R}^m$. Thus a development *unrolls* M_0 onto the Euclidean space $\mathbb{R}^m$. When $n = 3$ and $m = 2$ one can visualize M_0 as a twisted sheet of paper.

Hint: Part a) is not as innocent as it seems; skip this part if you get stuck. For b): Solutions of the ODE $\dot{X}(t) + \frac{\langle X(t), \ddot{u}(t)\rangle}{|\dot{u}(t)|^2}\dot{u}(t) = 0$ remain in $L(t)$. This can be used to construct a smooth family of orthonormal bases $X_1(t), \ldots, X_{m-1}(t)$ along $L(t)$. Use these and the implicit function theorem to construct charts for M_0. For d): Use the Gauss-Codazzi formula. For e): Recall that parallel transport intertwines along developements.

3. Prove that each of the following is a developable surface in $\mathbb{R}^3$.

a) A cone on an (embedded) curve $\Gamma \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ given by

$$M = \left\{ \lambda p + (1-\lambda) q \;\middle|\; \lambda < 1,\ q \in \Gamma \right\}, \quad p \in \mathbb{R}^3 \setminus \left(\mathbb{R}^2 \times \{0\} \right).$$

b) A cylinder on an (embedded) plane curve, i.e.

$$M = \left\{ q + tv \;\middle|\; t \in \mathbb{R},\ q \in \Gamma \right\},$$

where Γ are as in (i) and v is a fixed vector in $\mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\})$. (This is the cone with a point p at infinity).

c) The tangent developable to a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$

$$M := \left\{ \gamma(t) + s\dot{\gamma}(t) \;\middle|\; |t - t_0| < \epsilon,\ \ 0 < s < \epsilon \right\},$$

where $\dot{\gamma}(t_0)$ and $\ddot{\gamma}(t_0)$ are linearly independant and $\epsilon > 0$ is sufficiently small.

4. For $b \geq a > 0$ and $c \geq 0$ define

$$M_{a,b,c} := \{(u,v,w) \in \mathbb{C}^3 \,:\, |u| = a, |v| = b, w = cuv\}.$$

a) Prove that $M_{a,b,c}$ is diffeomorphic to the standard torus $S^1 \times S^1$. Also prove that $M_{a,b,c}$ is flat.

b) Show that the natural map $F : M_{a,b,c} \to M_{a',b',c'}$ defined by

$$(u,v,w) \mapsto (u',v',w') := \left(\frac{ua'}{a}, \frac{vb'}{b}, \frac{c'ua'vb'}{ab}\right)$$

is an isometry, exactly if $(a,b,c) = (a',b',c')$.

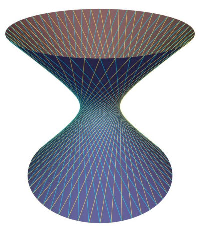
c) (**Challenge**) When are two flat tori $M_{a,b,c}$ and $M_{a',b',c'}$ isometric?

5. A 2-dimensional submanifold $M \subset \mathbb{R}^3$ is called a ruled surface if there is a straight line in M through every point. Every developable surface is ruled, however, there are ruled surfaces that are not developable. An example is the elliptic hyperboloid of one sheet depicted below.

$$M := \left\{(x,y,z) \in \mathbb{R}^3 \,:\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\right\}.$$

Prove that this manifold has negative Gaussian curvature and there are two straight lines through every point in M.

Hint: For ease of calculation, assume that $a = b$ in the calculation of curvature. Last sheet had a formula for curvature of surfaces of revolution.



6. Prove that the homotopy relation $\gamma_0 \sim \gamma_1$ is an equivalence relation on the space Ω_{pq} of curves with endpoints $p, q \in M$.