Exercise Sheet 13

Please hand in your solutions by December 18, 2017. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Show that the curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

is multi-linear over $C^\infty(M,\mathbb{R})$.

b) Show that the covariant derivative

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W$$

is multi-linear over $C^\infty(M,\mathbb{R})$.

c) Deduce that the covariant derivative $(\nabla_X R)(Y,Z)W(p)$ depends only on the values $X(p), Y(p), Z(p), W(p) \in T_pM$ and not on the particular vector fields.

Hint: In a) you have to prove that $R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = f(R(X,Y)Z)$ holds for all $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M,\mathbb{R})$. For this recall that $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X(fY) = (\mathcal{L}_X f)Y + f\nabla_X Y$. Part b) goes similar. The claim in c) is in fact equivalent to multi-linearity over $C^\infty(M,\mathbb{R})$.

2. Let $E_i = \frac{\partial \psi}{\partial x^i}$ for $i = 1,\dots,m$ and for $\psi : \Omega \to M$ a parametrisation. Take $\nabla_i R^\nu_{jk\ell}$ to be given by $\sum_{\nu=1}^m \nabla_i R^\nu_{jk\ell} E_\nu = (\nabla_{E_i} R)(E_j, E_k)E_\ell$. Prove that

$$\nabla_i R^\nu_{jk\ell} = \partial_i R^\nu_{jk\ell} + \sum_{\mu=1}^m \Gamma^\nu_{i\mu} R^\mu_{jk\ell} - \sum_{\mu=1}^m \left(\Gamma^\mu_{ij} R^\nu_{\mu k\ell} + \Gamma^\mu_{ik} R^\nu_{j\mu\ell} + \Gamma^\mu_{i\ell} R^\nu_{jk\mu} \right).$$

3. Prove the second Bianchi identity, i.e. prove that

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

4. Let M be a connected symmetric manifold.

a) Prove that M is complete.

b) For $p \in M$ denote by $\phi_p : M \to M$ the isometry with $\phi_p(p) = p$ and $d\phi_p(p) = -\mathbb{1}$. Prove that the map

$$\sigma : M \times M \to M, \qquad (p,q) \mapsto \phi_p(q)$$

is smooth.

c) Let γ be a non-constant geodesic. Prove that the map $\tau : (\mathbb{R}, +) \to (\mathrm{Isom}(M), \circ) : t \mapsto \tau_{\gamma,t} := \phi_{\gamma(t/2)} \circ \phi_{\gamma(0)}$ is a group homomorphism.

d) Prove that M is homogeneous.

Hint: Use Hopf-Rinow for b) and d), which is possible by a). In b), express $\phi_w(q)$ in terms of exponential maps and parallel transport for w in a geodesically convex neighbourhood of p. For c), see that τ is translation along the geodesic γ and use uniqueness of isometries. Additionally, the images of parallel vector fields come in handy in b) and c).

5. **a)** Show that S^n is a symmetric space. For this verify that the map $\phi_p : S^n \to S^n$ given by $\phi_p(x) = -x + 2\langle p, x\rangle p$ is an isometry satisfying $\phi_p(p) = p$ and $d\phi_p(p) = -\mathbb{1}$.

b) Show that a compact Lie subgroup $G \subset O(n)$ is a symmetric space. For this verify that the map $\phi_a : G \to G$ given by $\phi_a(g) = ag^{-1}a$ is an isometry satisfying $\phi_a(a) = a$ and $d\phi_a(a) = -\mathbb{1}$.

c) Let G be as in part b) above. Show that its sectional curvature is given by

$$K(\mathbb{1}, E) = \frac{1}{4}|[\xi, \eta]|^2$$

where E is a two dimensional subspace of $\mathfrak{g} = T_{\mathbb{1}}G$ and ξ, η an orthonormal basis for E.

Hint: For b) and c) recall exercise 2 of sheet 10 and exercise 6 of sheet 8. In c) the identity $\langle[\xi, \eta], \zeta\rangle = \langle\xi, [\eta, \zeta]$ for $\xi, \eta, \zeta \in T_{\mathbb{1}}G$ might be handy.

6. Prove that the sphere $S^n_r ;= \{x \in \mathbb{R}^{n+1} : |x| = r\}$ of radius $r > 0$ has sectional curvature $K(p, E) = \frac{1}{r^2}$.