Holiday Sheet 14

Our Christmas present to you are a lot of great exercises which prepare you well for the upcoming exam. No model solutions will be provided and you should not hand in your solutions. Have a nice festive season.

Merry Christmas And A Happy New Year !

1. **(Sectional curvature)** Let $p \in M \subset \mathbb{R}^n$ and let $E \subset T_pM$ be a two dimensional linear subspace. For $r > 0$ let L denote the ball of radius r in the $n - m + 2$ dimensional affine subspace of $\mathbb{R}^n$ through p and parallel to the vector subspace $E \oplus T_pM^{\perp}$:

$$L = \left\{ p + v + w \mid v \in E,\ w \in T_pM^{\perp},\ |v|^2 + |w|^2 < r^2 \right\}.$$

Show that, for r sufficiently small , $L \cap M$ is a $2-$dimensional manifold with Gauss curvature $K_{L\cap M}(p)$ at p given by

$$K_{L\cap M}(p) = K(p, E).$$

2. **(Sectional curvature in product manifolds)** Let $M_1 \subset \mathbb{R}^{\ell_1}$ and $M_2 \subset \mathbb{R}^{\ell_2}$ be two manifolds with product $M_1 \times M_2 \subset \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$. Show that the sectional curvatures

$$K_{(p_1,p_2)}\left((u_1, 0), (0, u_2)\right) := \frac{\langle R_{(p_1,p_2)}((u_1, 0), (0, u_2))(0, u_2), (0, u_1)\rangle}{|u_1|^2|u_2|^2} = 0$$

vanishes for all $u_i \in T_{p_i}M_i$, $i = 1, 2$.

3. (Derivation of the hyperbolic metric) The hyperbolic metric g on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ is given by

$$g_z(\xi_1, \xi_2) := \frac{4\langle \xi_1, \xi_2 \rangle}{(1 - |z|^2)^2}, \quad \xi_i \in T_z\mathbb{D} \cong \mathbb{C}, \quad i = 1, 2, \quad z \in \mathbb{D}. \tag{1}$$

We want to show that the metric (1) is up to multiplication by positive constant characterised by the following:

Let $S_R \subset \mathbb{C}$ be a circle orthogonal to $S^1 = \partial\mathbb{D}$. The restriction of the inversion with respect to S_k is an isometry of $\mathbb{D}$.

Recall that the inversion in the circle $S_R = \{z \in \mathbb{C} : \ |z - z_0| = R\}$ is the map

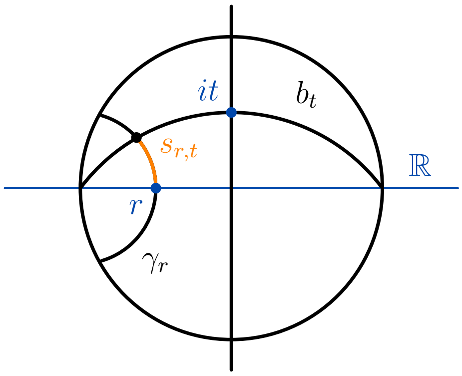
$$\mathrm{Inv} : \mathbb{C} \to \mathbb{C} : z \mapsto \mathrm{Inv}(z) = z' = z_0 + \frac{R^2(z - z_0)}{|z - z_0|^2},$$

where z' is the unique point on the line through z_0 and z with the property $|z - z_0||z' - z_0| = R^2$.

a) Show that there exists a smooth function $f : [0, 1) \to \mathbb{R}$ such that

$$g_z(\xi_1, \xi_2) = f(|z|)^2 \langle \xi_1, \xi_2 \rangle, \qquad \text{for all } z \in \mathbb{D}, \ \xi_1, \xi_2 \in \mathbb{C}.$$

b) We want to determine the function f now. For $-1 < t < 1$, let b_t be the arc through the points -1, it, 1. Let γ_r, for $-1 < r < 1$, be the arc which is orthogonal to $S^1 = \partial\mathbb{D}$ and $\mathbb{R}$ and goes through the point $r \in \mathbb{R}$. Finally let $s_{t,r}$ be the piece of γ_r which is between $\mathbb{R}$ and b_t.



Prove that

$$L(s_{r,t}) = L(s_{0,t}),$$

where $L(s_{r,t})$ denotes the length of $s_{r,t}$ with respect to the metric g.

c) Show that

$$f(r) = \lim_{t \to 0} \frac{L(s_{r,t})}{L^E(s_{r,t})}, \tag{2}$$

where L^E denotes the Euclidean length. Determine f in (2) which depends only on $c_0 := \frac{d}{dt}\Big|_{t=0} L(s_{0,t}) \in \mathbb{R}$.

4. (The Poincare Disc model) Recall that hyperbolic space is defined as the set

$$\mathbb{H}^m := \{x \in \mathbb{R}^{m+1} \mid -x_0^2 + x_1^2 + \cdots + x_m^2 = -1,\, x_0 > 0\}$$

and with first fundamental form obtained by restricting the indefinit quadratic form

$$Q : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}, \qquad Q(x, y) := -x_0 y_0 + x_1 y_1 + \cdots + x_m y_m$$

to the tangent spaces of $\mathbb{H}^m$.

a) Show that the stereographic from $(-1, 0) \in \mathbb{R}^{m+1}$ defines a diffeomorphism from $\{0\} \times \mathbb{D}^m$ to $\mathbb{H}^m$ given by the formula

$$\mathbb{D}^m \to \mathbb{H}^m : \qquad y \mapsto \left(\frac{1 + |y|^2}{1 - |y|^2}, \frac{2y}{1 - |y|^2} \right)$$

b) Show that the pullback of the first fundamental form of $\mathbb{H}^m$ to $\mathbb{D}^m$ yields the metric tensor

$$g_{ij}(y) = \frac{4\delta_{ij}}{(1 - |y|^2)^2}.$$

5. (Hyperbolic metric of the upper half plane)

Let $\mathbb{H} := \{x + \mathbf{i}y \in \mathbb{C} \mid y > 0\} \subset \mathbb{C}$ be the upper half plane.

a) Prove that the map

$$\psi : \mathbb{D} \to \mathbb{H}, \qquad \psi(z) = \frac{z + \mathbf{i}}{\mathbf{i}z + 1}$$

is a (biholomorphic) diffeomorphism, where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$ denotes the unit disc.

b) Denote by g the hyperbolic metric on $\mathbb{D}$ obtained in the previous exercise. Prove that

$$\psi_* g_z := (\psi^{-1})^* g_z = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

where $z = x + \mathbf{i}y$.

c) The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ through the Möbius transformations

$$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}$$

Show that the Möbius transformations act via isometries for the hyperbolic metric $(\psi^{-1})^* g$. Determine the stabilizer of $\mathbf{i} \in \mathbb{H}$.

6. (Further models of the hyperbolic plane) Denote by $\mathrm{Sym}(2,\mathbb{R}) \subset \mathbb{R}^{2\times 2}$ the space of real symmetric 2×2 matrices. The groups $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathrm{Sym}(2,\mathbb{R})$ via

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{Sym}(2,\mathbb{R}) \to \mathrm{Sym}(2,\mathbb{R}), \qquad A_* S := ASA^T.$$

Let P_2 be the orbit of the unit matrix $\mathbb{1}$. It is known from linear algebra that the orbit

$$P_2 = SL_2(\mathbb{R}) \cdot \mathbb{1} \subset \mathrm{Sym}(2,\mathbb{R})$$

contains all positive symmetric matrices with determinant is 1.

a) Show that P_2 is a 2-dimensional manifold.

b) Define a Riemannian metric on P_2 by

$$g_S : T_S P_2 \oplus T_S P_2 \to \mathbb{R}, \qquad g_S(X,Y) := \mathrm{tr}(S^{-1}XS^{-1}Y)$$

for $S \in P_2$. Show that $\mathrm{SL}_2(\mathbb{R})$ acts by isometries on (P_2, g).

c) Show that (P_2, d) (equipped with the intrinsic distance induced by g_S) is isometric to the hyperbolic plane.

Hint: In c) use the upper half plane model for hyperbolic space. It suffices to construct a $\mathrm{SL}_2(\mathbb{R})$ equivariant map $P_2 \to \mathbb{H}$ which intertwines the Riemannian metrics.

7. (Determinant of the Gauss map) Let $M \subset \mathbb{R}^{m+1}$ be a m-dimensional manifold. We assume that there is a smooth map

$$\nu : M \to S^m$$

such that $\nu(p) \perp T_pM$ for every $p \in M$. Differentiating this map we obtain a linear map

$$d\nu(p) : T_pM \to T_{\nu(p)}S^m = T_pM.$$

Define

$$k(p) := \det\left(d\nu(p)\right).$$

We investigate in this exercise weather k is an intrinsic invariant of M. Here intrinsic means that k depends only on the first fundamental form of M and so we prove an extension of the Theorema Egregium.

a) Show that k is intrinsic when the dimension $m = 2\ell$ is even.

b) Show that the absolut value $|k|$ is intrinsic when $m \geq 3$ and m is odd.

c) What happens when $M \subset \mathbb{R}^2$ is a curve?

Hint: For a) and b): Try to express $k(p)$ in terms of the metric tensor g_p and the curvature endomorphisms $R_p(u,v) : T_pM \to T_pM$ with $u, v \in T_pM$. For this choose a orthonormal basis $e_1, \dots, e_m$ of T_pM and show that the sectional curvature $K_p(e_i, e_j)$ can be expressed as the determinant of a 2×2 minor of $d\nu(p)$.

8. (Totally geodesic submanifolds)

a) Let M be a Riemannian manifold and $L \subset M$ be a submanifold. Prove that the following are equivalent.

(i) If $\gamma : I \to M$ is a geodesic on an open intervall I with $\gamma(0) \in L$ and $\dot\gamma(0) \in T_{\gamma(0)}L$, then there exists $\epsilon > 0$ such that $\gamma(t) \in L$ for $|t| < \epsilon$.

(ii) If $\gamma : I \to M$ is a smooth curve on an open intervall I and Φ_γ denotes the parallel transport along γ. Then

$$\Phi_\gamma(t,s)T_{\gamma(s)}L \subset T_{\gamma(t)}L \qquad \text{for all } t, s \in I.$$

(iii) If $\gamma : I \to M$ is a smooth curve on an open intervall I and $X \in \mathrm{Vect}(\gamma)$ with $X(t) \in T_{\gamma(t)}L$ for all $t \in I$, then $\nabla X(t) \in T_{\gamma(t)}L$ as well.

A submanifold that satisfies these equivalent conditions is called **totally geodesic**.

b) Denote by P the space of positive symmetric matrices and by $P_0 \subset P$ the submanifold of matrices with determinant 1. Show that $P_0 \subset P$ is totally geodesic.

9. Let $p \in M \subset \mathbb{R}^n$ and take an orthonormal basis $\{e_i\}_{i=1,\dots,m} \subset T_pM$. Define

$$\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R} : \qquad (u,v) \mapsto \sum_{i=1}^m \langle R(e_i,u)v, e_i\rangle.$$

a) Prove that the map R_p is independent of the choice of orthonormal basis.

b) Prove that the **Ricci curvature tensor**

$$\mathrm{Ric} : \mathrm{Vect}(M)\times\mathrm{Vect}(M) \to C^\infty(M,\mathbb{R}) : (X,Y) \mapsto \big(p \mapsto \mathrm{Ric}_p(X(p),Y(p))\big)$$

is well-defined, symmetric and linear over $\mathcal{F}(M)$.

c) Prove that for $m = 2$, we get

$$\mathrm{Ric}(X,Y) = K\langle X,Y\rangle$$

for all $X, Y \in \mathrm{Vect}(M)$ where K is the Gauss curvature.

Hint: For b), use the existence of local sections of the orthonormal frame bundle $\mathcal{O}(M)$.

10. Take $\gamma : I = [a,b] \to M \subset \mathbb{R}^n$ to be a unit-speed geodesic and a vector field $X = \frac{d}{ds}\big|_{s=0}\gamma_s \in \mathrm{Vect}(\gamma)$ along γ with $X(a) = X(b) = 0$. Prove that the **second variation of length** is given by the formula

$$\frac{d^2}{ds^2}\bigg|_{s=0} L(\gamma_s) = \int_a^b \big(|\nabla_t X| - R(X,\dot\gamma,\dot\gamma,X)\big).$$

11. (**Bonnet-Myers Theorem**) Let $M \subset \mathbb{R}^n$ be a complete manifold of dimension $m \geq 2$ such that there is $k > 0$ with

$$\mathrm{Ric}(X, X) \geq (m-1)k|X|^2$$

for all $X \in \mathrm{Vect}(M)$ where Ricci curvature was defined in Exercise 9. Then the diameter

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{k}}$$

is bounded and so in particular M is compact.

Hint: We give a series of hints to guide you through the proof of this classical theorem.

(A) Apply Hopf-Rinow to any two points $p, q \in M$ to get a unit speed geodesic $\gamma : [0, \ell] \to M$. Our goal is to bound ℓ.

(B) Use a parallel orthonormal frame $\{X_i\}_{i=1,\dots,m} \subset \mathrm{Vect}(\gamma)$ with $X_1 = \dot{\gamma}$ and define $\{Y_i\}_{i=2,\dots,m} \subset \mathrm{Vect}(\gamma)$ given by $Y_i(t) = \cos(\frac{\pi t}{\ell}) X_i(t)$.

(C) Why is the second variation of length from Exercise 10. non negative? Plug Y_i into the formula.

(D) Find the Ricci curvature and conclude.

12. Let M be a compact smooth manifold. A free homotopy class of free loops is a connected component of the space $C^\infty(M, S^1)$. More precisely, we say that two smooth loops

$$\gamma_0 : S^1 \to M, \qquad \gamma_1 : S^1 \to M$$

are freely homotopic, if and only if there exists a smooth map $\gamma : S^1 \times [0,1] \to M$ such that $\gamma_0 = \gamma(\cdot, 0)$ and $\gamma_1 = \gamma(\cdot, 1)$. The aim of this exercise is to proof the following result:

Every free homotopy class of free loops contains a smooth geodesic.

You may use the following fact in your solution without further justification:
Fact: Suppose γ_0 and γ_1 are smooth loops and there exists a continuous homotopy $\gamma : S^1 \times [0,1] \to M$ with $\gamma_0 = \gamma(\cdot, 0)$ and $\gamma_1 = \gamma(\cdot, 1)$. Then there exists also a smooth homotopy $\tilde{\gamma} : S^1 \times [0,1] \to M$ such that $\gamma_0 = \tilde{\gamma}(\cdot, 0)$ and $\gamma_1 = \tilde{\gamma}(\cdot, 1)$.

a) Fix a free homotopy class of free loops $\pi \subset C^\infty(S^1, M)$. Choose a minimising sequence of curves $\{\gamma_k\}_{k\in\mathbb{N}} \subset \pi$ such that $|\dot{\gamma}_k|$ is constant for each k and

$$\lim_{k\to\infty} L(\gamma_k) = \inf_{\gamma\in\pi} L(\gamma).$$

Use Arzela-Ascoli to show that there exists a continuous curve $c : S^1 \to M$ and that γ_k converges uniformly to c.

b) Let $0 < \epsilon < \mathrm{inj}(M)$ be given. Show that there exists a finite partition $0 = t^0 < t^1 < \cdots < t^N = 2\pi$ such that

$$d_M(p_k^j, p_k^{j+1}) < \epsilon, \qquad \text{for all } k \in \mathbb{N} \text{ and } j = 0, \dots, N-1$$

where $p_k^j := \gamma_k(e^{\mathbf{i}t_j})$.

c) Define $v_k^j \in T_{p_k^j}M$ by the equation $\exp_{p_k^j} v_k^j = p_k^{j+1}$. Show that (after passing to a subsequence) the following limits exists

$$\lim_{k\to\infty} p_k^j =: p^j = c(t^j), \qquad \lim_{k\to\infty} v_k^j =: v^j \in T_{p^j}M.$$

d) Denote by $\tilde{\gamma}_k : S^1 \to M$ the piecewise smooth geodesic going through the points $\{p_k^j\}_{j=1}^N$ with velocity $\{v_k^j\}_{j=1}^N$ and let $\tilde{c} : S^1 \to M$ be the piecewise smooth geodesic going through the points $\{p^j\}_{j=1}^N$ with velocity $\{v^j\}_{j=1}^N$. Show that $\tilde{\gamma}_k$ converges on each intervall in $C^\infty(e^{\mathbf{i}[t^j,t^{j+1}]}, M)$ to $\tilde{c}$. Deduce from this that

$$L(\tilde{c}) = \lim_{k\to\infty} L(\tilde{\gamma}_k) \leq \lim_{k\to\infty} L(\gamma_k) = \inf_{\gamma\in\pi} L(\gamma)$$

e) Show that $\tilde{c}$ is a smooth geodesic and $\tilde{c} \in \pi$.

Hint: In *b*): Any sufficiently fine partition will do the job. More precisely, use uniform continuity of c and that γ_k converges uniformly to c. In *e*): Argue by contradiction and use the curve shortening Lemma to show smoothness of $\tilde{c}$. The final claim $\tilde{c} \in \pi$ uses the fact stated in the beginning and that connected components of $C^0(S^1, M)$ are open.

13. (Tschebyscheff net)

Let $M \subset \mathbb{R}^3$ be a 2-dimensional manifold, $\Omega := (0, A) \times (0, B) \subset \mathbb{R}^2$ be an open rectangle and let

$$\psi : \Omega \to M \subset \mathbb{R}^3$$

be a smooth parametrisation. Define $g_{ij} : \Omega \to \mathbb{R}$ by $g_{ij} := \langle \partial_1\psi, \partial_2\psi \rangle$.

a) Show that the following are equivalent:

(i) For each rectangle $R = [u_1, u_1 + a] \times [u_2, u_2 + b] \subset \Omega$ opposite sides of $\psi(R)$ have the same length.

(ii) The following relation holds: $\partial_2 g_{11} = 0 = \partial_1 g_{22}$

b) Show that for a parametrisation ψ as in part a) there exists diffeomorphism $f : \Omega \to \tilde{\Omega}$ between open subsets of $\mathbb{R}^2$ such that the first fundamental form of $\tilde{\psi} = \psi \circ f^{-1} : \tilde{\Omega} \to M$ is given by

$$\tilde{g}_{ij} = \begin{pmatrix} 1 & \cos w \\ \cos w & 1 \end{pmatrix},$$

Geometrically this means that the coordinate vector fields for this parametrisation are of unit length and $\omega = \omega(u_1, u_2)$ is the angle between the coordinate lines.