Solution 1

1. **a)** Show that a subset $M \subset \mathbb{R}^k$ is a 0−dimensional submanifold if and only if M is discrete, i.e. for every $p \in M$ there is an open set $U \subset \mathbb{R}^k$ such that $U \cap M = \{p\}$.

b) Show that a subset $M \subset \mathbb{R}^m$ is a m−dimensional submanifold if and only if M is open.

c) If $M_i \subset \mathbb{R}^{k_i}$ is a m_i-manifold for $i = 1, 2$ show that $M_1 \times M_2$ is a $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{k_1+k_2}$. Prove by induction that the n−torus $\mathbb{T}^n$ is a smooth submanifold of $\mathbb{C}^n$.

Solution:

a) For the first implication ⇒, we use the definition of manifold for $m = 0$. Fix $p \in M$ and then there is an open neighbourhood $U \subset \mathbb{R}^k$ of p such that $U \cap M$ is diffeomorphic to an open set $\Omega \subset \mathbb{R}^0 = \{*\}$. This means that $\Omega = \{*\}$ and $\psi : \{*\} \to U \cap M$ is a bijection. So $M \cap U = \{p\}$ and because p was arbitrary, M is discrete.

For the converse, for any $p \in M$, take an open neighbourhood $U \cap M = \{p\}$ and take charts $\varphi : U \cap M \to \mathbb{R}^0 : p \mapsto *$.

b) For the first implication ⇒, we use Theorem 1. This theorem gives us that every $p \in M$ has an open neighbourhood U and there is a function $f : U \to \mathbb{R}^0$ such that $M \cap U = f^{-1}(*)$. Therefore we have that $U = f^{-1}(*) = M \cap U$ and so M is the union of open sets, and thereby open. (The inverse theorem is built into Theorem 1. You can also prove the statement directly from the inverse function theorem.)

For the converse, if M is open, use one chart $\varphi : M \to M : x \mapsto x$.

c) If for $i = 1, 2$ $\varphi_i : M \cap U_i \to \Omega_i \subset \mathbb{R}^{m_i}$ is a chart around $p_i \in M_i$, then $\varphi_1 \times \varphi_2 : (U_1 \times U_2) \cap (M_1 \times M_2) \to \Omega_1 \times \Omega_2 \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ fulfils all the conditions for chart of $M_1 \times M_2$ around $(p_1, p_2) \in M_1 \times M_2$. So $M_1 \times M_2$ is an $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{k_1+k_2}$.

For the torus, we have that $\mathbb{T}^n = \mathbb{R}^n \backslash \mathbb{Z}^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$, where S^1 is the 1-dimensional sphere as in Exercise 3.

2. Consider the subset $L = \{(x, y) \in \mathbb{R}^2 : xy = 0\} \subset \mathbb{R}^2$.

a) Prove that L is not a 0−dimensional submanifold of $\mathbb{R}^2$.

b) Prove that L is not a 1−dimensional submanifold of $\mathbb{R}^2$.

c) Prove that L is not a 2−dimensional submanifold of $\mathbb{R}^2$.

Hint: Use Exercise 1 in a) and c). Use Theorem 1 from the lecture in b).
Find another proof in b) using the connected components of the complement of a point.

Solution:

a) The union of the x-axis and y-axis in $\mathbb{R}^2$ is not discrete, so no 0-manifold.

b) A first possible solution is using Theorem 1. Assume L is a 1-manifold. At the point 0, we need to have an open set U and a function $f : U \to \mathbb{R}$ such that $f^{-1}(0) = L \cap U$. U contains a ball of a small radius ϵ and so f is a smooth function such that $f(t\epsilon, 0) = f(0, t\epsilon) = 0$ for all $t \in (-1, 1)$. So $df(0) = 0$ cannot be surjective, and this is a contradiction. A second possibility is to use the definition of the manifold directly, and see that a chart $\varphi : U \cap L \to (a,b) \subset \mathbb{R}$ around 0 is in particular a homeomorphism. So it carries connected components to connected components. Thus $U \cap L \setminus \{0\}$ and $(a,b) \setminus \{\varphi(0)\}$ must have the same number of connected components. Now $U \cap L \setminus \{0\}$ has four connected components and $(a,b) \setminus \{\varphi(0)\}$ has two, which again implies a contradiction.

c) The union of the x-axis and y-axis in $\mathbb{R}^2$ is not open, so no 2-manifold.

3. **a)** Show that $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$ is a n-dimensional submanifold of $\mathbb{R}^{n+1}$ by using Theorem 1.

b) Write down a small collection of charts $\varphi_i : U_i \cap S^n \to V_i \subset \mathbb{R}^n$ such that $\bigcup_{1\leq i\leq N} V_i = S^n$. What is the smallest N?

Hint: Try stereographic projections from North and South poles.

c) For a fixed pair (i,j) with $i \neq j$ calculate the *change of chart*

$$\varphi_{ji} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j \cap S^n) \to \varphi_j(U_i \cap U_j \cap S^n).$$

Solution:

a) Define $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by $x \mapsto \sum_{i=1}^{n+1} x_i^2$. Then $df(x)\hat{x} = 2x^\top \hat{x}$ and for any point $x \in S^n$, there is at least one $x_i \neq 0$ and so $df(x)e_i \neq 0$ where e_i is the i-th base vector of $\mathbb{R}^{n+1}$. Therefore by Theorem 1, $S^n = f^{-1}(1)$ is a submanifold of dimension $n = (n+1) - 1$.

b) The stereographic projection φ_1 from the North pole $p_N = (0,\dots,0,1)$ of a point $x \in V_1 := S^n \setminus p_N$ is equal to the point of intersection the line through p_N and x with the hyperplane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{R}^n$. A small calculation yields

$$\varphi_1(x) = \frac{1}{1-x_n} x.$$

The second chart will be the stereographic projection from the south pole $p_S = (0,\dots,0,-1)$

$$\varphi_2 : V_2 := S^n \setminus \{p_S\} \to \mathbb{R}^n : x \mapsto \frac{1}{1+x_n} x.$$

Such a collection of charts covering the whole manifold is called an atlas. We thus have $N = 2$ and this is the minimal number as a compact space (S^n) cannot be homeomorphic to a non-compact space (open subset of $\mathbb{R}^n$).

c) Let us take the pair $(1,2)$. Thus, we first need to invert the stereographic projection. For this, we calculate the intersection of the line connecting p_N with $(y_1,\dots,y_n,0)$ with the sphere S^n. A small calculation gives

$$\varphi_1^{-1}(y) = \left(\frac{2y_1}{|y|^2+1}, \dots, \frac{2y_n}{|y|^2+1}, \frac{|y|^2-1}{|y|^2+1} \right).$$

So the change of charts is given by

$$\varphi_2 \circ \varphi_1^{-1}(y) = \frac{y}{|y|^2} \text{ for } y \in \mathbb{R}^n.$$

These formulae can also be derived by using similar triangles.

4. **a)** Show that the general linear group

$$GL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

b) Show that the special linear group

$$SL(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

c) Show that the orthogonal group[1]

$$O(n) = \{A \in \mathbb{R}^{n\times n} : A^\top A = \mathbb{1}\}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

d) Show that the symplectic group[2]

$$Sp(2n,\mathbb{R}) = \{A \in \mathbb{R}^{2n\times 2n} : A^\top J_0 A = J_0\} \text{ with } J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$$

is a submanifold of $\mathbb{R}^{n\times n}$. What is its dimension?

Hint: Use Exercise 1 for a). Use Theorem 1 with appropriate target spaces (a subset of all matrices) in b), c) and d). Jacobi's formula is handy in b).

Solution:

a) As $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous and $\mathbb{R}\setminus\{0\}$ is open, $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$ is open in $\mathbb{R}^{n\times n}$ and so by Exercise 1 a n^2 dimensional submanifold of $\mathbb{R}^{n\times n}$.

b) We want to apply Theorem 1 with $f = \det : \mathbb{R}^{n\times n} \to \mathbb{R}$. Therefore, we need to see that for every $A \in SL(n,\mathbb{R})$, $d\det(A)$ is surjective. By Jacobi's formula, we have

$$d\det(A)\hat{A} = \det(A)\operatorname{tr}(A^{-1}\hat{A}) = \operatorname{tr}(A^{-1}\hat{A}) \quad \text{for } \hat{A} \in \mathbb{R}^{n\times n}.$$

So for $\hat{A} = A$, we get $d\det(A)A = n \neq 0$ and so Theorem 1 applies, proving that $SL(n,\mathbb{R})$ is a $n^2 - 1$ dimensional submanifold.

c) We see that $A^\top A$ is symmetric, so we consider $f : \mathbb{R}^{n\times n} \to Symm(n) \cong \mathbb{R}^{n(n+1)/2}$ where $Symm(n)$ stands for symmetric matrices. We want to apply Theorem 1 again. Therefore, we calculate the differential of f as

$$df(A)\hat{A} = \hat{A}^\top A + A^\top \hat{A} \quad \text{for } \hat{A} \in \mathbb{R}^{n\times n}.$$

Now for any S symmetric, we need to prove that there is $\hat{A}$ such that $df(A)\hat{A} = S$ under the assumption $A^\top A = \mathbb{1}$. One can see that $\hat{A} = \frac{1}{2}AS$ does the job. So $O(n)$ is a $n^2 - n(n+1)/2 = n(n-1)/2$ dimensional submanifold of $\mathbb{R}^{n\times n}$.

d) As in the previous tasks, we want to apply Theorem 1. We note that $A^\top J_0 A$ is anti-symmetric and so we look at $f : \mathbb{R}^{2n\times 2n} \to Asymm(2n) \cong \mathbb{R}^{2n(2n-1)/2}$ where $Asymm(2n)$ is the set on anti-symmetric matrices. The differential of f is given by

$$df(A)\hat{A} = \hat{A}^\top J_0 A + A^\top J_0 \hat{A} \quad \text{for } \hat{A} \in \mathbb{R}^{n\times n}.$$

Using the relation $J_0^2 = -\mathbb{1}$, we get for $\hat{A} = -\frac{1}{2}AJ_0M$ with M anti-symmetric that $df(A)\hat{A} = M$. So $Sp(2n,\mathbb{R})$ is a $4n^2 - 2n(2n-1)/2 = 2n^2 + n$ dimensional submanifold of $\mathbb{R}^{2n\times 2n}$.

[1]This group is important for Riemannian geometry.
[2]This group is important for symplectic geometry.

5. Let $M \subset \mathbb{R}^k$ be a submanifold of $\mathbb{R}^k$ and define the distance function d_0 on M by $d_0(p,q) = |p-q|$ for $p, q \in M$. Show that the topology on M induced by d_0 is the same topology as the subspace topology induced on M as a subset of $\mathbb{R}^n$. Recall that the subspace topology is defined by $V \subset M$ is open exactly if there is an open set $U \subset \mathbb{R}^k$ such that $V = U \cap M$.

Solution: We need to see that the topologies agree. Let us start with an open set V of the relative topology. This means there is U open subset of $\mathbb{R}^k$ such that $V = U \cap M$. So every point $p \in V$ is in U. Thus there is an Euclidean ball $B_r(p)$ contained in U and $B_r^{d_0}(p) = B_r(p) \cap M \subset U \cap M = V$. Therefore, V is also open in the metric topology with respect to d_0. For the converse, take V open in the d_0-metric topology. This means there is a collection $\{(p_i, r_i)\}_{i \in I}$ of pairs of points p_i of V and radii r_i such that $V = \bigcup_{i \in I} B_{r_i}^{d_0}(p_i)$. Thus we take the union of Euclidean balls $U = \bigcup_{i \in I} B_{r_i}(p_i)$ which is open in $\mathbb{R}^k$ and so $V = U \cap M$ is open with respect to the subspace topology.

6. **a)** Prove that path connected topological spaces are connected.

b) Prove that any connected submanifold $M \subset \mathbb{R}^k$ is also path connected.

Hint: Prove first that any submanifold is locally path connected. I.e. for every $p \in M$ there is an open set $p \in V \subset M$ such that V is path connected.

Solution:

a) Suppose not. That is assume X (a topological space) is path-connected which is not connected. Then there would be U_1, U_2 non-empty open subsets of X such that $X = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Take $p_i \in U_i$ for $i = 1, 2$ and connect them by a path $\gamma_{12} : [0,1] \to X$. Then $V_i := \gamma_{12}^{-1}(U_i)$ are non-empty, open sets of $[0,1]$ such that $[0,1] = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. However, $[0,1]$ is connected, thereby leading to a contradiction.

b) The first step is to prove as in the hint, that M is locally path connected. For this take for $p \in M$, a chart $\varphi : U \cap M \to \Omega \subset \mathbb{R}^m$ and take a Euclidean ball $B_\epsilon(\varphi(p)) \subset \Omega$ which exists, because $\Omega \subset \mathbb{R}^m$ is open. Such a ball is open and path-connected, and so is $\varphi^{-1}(B_\epsilon(\varphi(p))) \subset M$, because φ is a homeomorphism.

So we reduce the statement of this exercise to the more general statement

$$X \text{ is path-connected } \Leftrightarrow X \text{ is connected and locally path-connected.}$$

The first implication was already proven in a). For the converse, take $p \in X$ and define $U = \{q \in X : q \text{ can be connected to } p \text{ by a path.}\}$. Then we start by proving that U is open. Indeed, for $q \in U$ there is a path γ_{pq} from p to q and there is by local path-connectedness an open set V containing q such that any point $r \in V$ is connected to q by a path γ_{qr}. Then r is connected to p by the concatenated path $\gamma_{qr} \# \gamma_{pq}(t) := \begin{cases} \gamma_{pq}(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma_{qr}(2t-1) & \text{for } t \in (\frac{1}{2}, 1] \end{cases}$ and so $V \subset U$. Thus U is open.

On the other hand, we prove that U is closed. Indeed, take a point q in the closure of U and there is by local path-connectedness an open set V containing q such that any point $r \in V$ is connected to q by a path γ_{rq}. By definition of closure, $V \cap U \neq \emptyset$ and for any $r \in V \cap U$ there is a path γ_{pr}. Thus q is connected to p by the concatenated path $\gamma_{rq} \# \gamma_{pr}$ and thereby $q \in U$. Hence, U is open and closed, and therefore $U = X$ as X is connected. Hence, X is path-connected, by concatenating paths from and to p.