Solution 2

1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (kinetic plus potential energy) by $H(x, y) := \frac{1}{2}|y|^2 + V(x)$. Prove that c is a regular value of H if and only if it is a regular value of V.

Solution: Assume c is a regular value of H. This means that for every $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $H(x, y) = c$ the differential $dH(x, y)$ given by

$$dH(x, y)[\hat{x}, \hat{y}] = y^\top \hat{y} + dV(x)\hat{x}$$

is surjective. So in particular, the differential $dH(x, 0)[\hat{x}, \hat{y}] = dV(x)[\hat{x}]$ is surjective, for all $x \in V^{-1}(c)$. I.e. c is a regular value of V.

For the converse, assume c is a regular value for V. Then as before, $dH(x, 0)$ is already surjective for all $(x, 0) \in H^{-1}(c)$. On the other hand, for all points $(x, y) \in H^{-1}(c)$ with $y \neq 0$, we see that $dH(x, y)[0, y] = |y|^2 \neq 0$, and so c is also a regular value of H.

2.

a) Given an open set $U \subset \mathbb{R}^k$. What are its tangent spaces?

b) Given two submanifolds M_1, M_2, what are the tangent spaces of the product manifold $M_1 \times M_2$?

c) Given $U \subset \mathbb{R}^k$ open and a smooth function $f : U \to \mathbb{R}^\ell$. What are the tangent spaces of $\mathrm{graph}(f) := \{(x, y) \in \mathbb{R}^{k+\ell} : x \in U, y = f(x)\}$?

d) What are the tangent space of the sphere S^n ?

e) What is the tangent space $T_{\mathbb{1}} SL(n, \mathbb{R})$?

f) What is the tangent space $T_{\mathbb{1}} O(n)$?

Hint: For a), b), d), e), look back at ExSheet 1 Ex 1 c), Ex 3 and Ex 4 b), c). For c), example 1.2.3. of the lecture notes might come in handy. Use the Theorem from the lecture on tangent spaces in the different settings.

Solution:

a) A bit of a tautology. $T_x U = \mathbb{R}^k$ for all $x \in U$.

b) For $(p_1, p_2) \in M_1 \times M_2$, take parametrisations $\psi_i : \Omega_i \subset \mathbb{R}^{m_i} \to U_i \cap M_i \subset \mathbb{R}^{k_i}$ around $p_i = \psi_i(0)$ for $i = 1, 2$ and know from last exercise sheet that $\psi_1 \times \psi_2$ is a parametrisation for $M_1 \times M_2$ around (p_1, p_2). So we have by the theorem on tangent spaces from the lecture, that

$$\begin{aligned} T_{(p_1,p_2)} M_1 \times M_2 &= \mathrm{image}(d(\psi_1 \times \psi_2)(0) : \mathbb{R}^{m_1+m_2} \to \mathbb{R}^{k_1+k_2}) \\ &= \mathrm{image}(d(\psi_1)(0) : \mathbb{R}^{m_1} \to \mathbb{R}^{k_1}) \times \mathrm{image}(d(\psi_2)(0) : \mathbb{R}^{m_2} \to \mathbb{R}^{k_2}) \\ &= T_{p_1} M_1 \times T_{p_2} M_2. \end{aligned}$$

c) A parametrisation of $\mathrm{graph}(f)$ is given by $\psi : U \subset \mathbb{R}^k \to \mathrm{graph}(f) \subset \mathbb{R}^{k+\ell} : x \mapsto (x, f(x))$. So the tangent space is given by

$$T_{(x,f(x))}\, \mathrm{graph}(f) = \mathrm{image}(d\psi(x) : \mathbb{R}^k \to \mathbb{R}^{k+\ell}) = \mathrm{graph}(df(x)) \text{ for } x \in U.$$

d) Recall $S^n = f^{-1}(0)$ for $f : \mathbb{R}^{n+1} \to \mathbb{R} : x \to |x|^2 - 1$ and 0 is a regular value of f. By the theorem on tangent spaces, we have that for $v \in S^n$,

$$T_v S^n = \ker df(v) = \{\hat{x} \in \mathbb{R}^n : df(v)\hat{x} = 2v^\top \hat{x} = 0\} = (\mathrm{span}(v))^\perp.$$

e) Recall $SL(n,\mathbb{R}) = \det^{-1}(1)$ and 1 is a regular value for det. Also recall that $d\det(A)\hat{A} = \det(A)\operatorname{tr}(A^{-1}\hat{A})$ for $A \in GL(n,\mathbb{R})$. Thus the tangent space at $\mathbb{1}$ is

$$T_{\mathbb{1}}SL(n,\mathbb{R}) = \ker(d\det(\mathbb{1})) = \{\hat{A} \in \mathbb{R}^{n\times n} : \operatorname{tr}(\hat{A}) = 0\},$$

the space of traceless matrices.

f) Recall $O(n) = f^{-1}(\mathbb{1})$ for $f : \mathbb{R}^{n\times n} \to \mathrm{Symm}(n) \cong \mathbb{R}^{n(n+1)/2} : A \to A^\top A$ and $\mathbb{1}$ is a regular value of f. Also recall that $df(A)\hat{A} = \hat{A}^\top A + A^\top \hat{A}$. Thus the tangent space at $\mathbb{1}$ is

$$T_{\mathbb{1}}O(n) = \ker(df(\mathbb{1})) = \{\hat{A} \in \mathbb{R}^{n\times n} : \hat{A}^\top + \hat{A} = 0\},$$

the space of anti-symmetric matrices.

3. Show that $K := \{(x,y,z) \in \mathbb{C}^3 : x^4 + y^4 + z^4 + 1 = 0\} \subset \mathbb{C}^3 \cong \mathbb{R}^6$ is a smooth manifold and write down its tangent spaces. What is the dimension of K?

K is called a K3 surface and is an important object of study in complex algebraic geometry.

Hint: Get real, and you shall be caught up in a huge mess. Continue to use complex variables. Just use Theorem 1 and remember holomorphic means complex linear differential.

Solution:

- K is a smooth manifold. Take $f : \mathbb{C}^3 \cong \mathbb{R}^6 \to \mathbb{C} \cong \mathbb{R}^2 : (x,y,z) \mapsto x^4 + y^4 + z^4$ and see that $K = f^{-1}(-1)$. So let us calculate the differential of f

$$df(x,y,z)[\hat{x},\hat{y},\hat{z}] = 4x^3\hat{x} + 4y^3\hat{y} + 4z^3\hat{z}.$$

For $\lambda \in \mathbb{C}$ and $(x,y,z) \in K$, put $(\hat{x},\hat{y},\hat{z}) = -\frac{\lambda}{4}(x,y,z)$ and see that $df(x,y,z)[\hat{x},\hat{y},\hat{z}] = \lambda$. Thus -1 is a regular value and thereby K is a $6-2=4$-dimensional submanifold of $\mathbb{R}^6$.

- Tangent spaces of K: For $(x,y,z) \in K$, we have

$$T_{(x,y,z)}K = \ker df(x) = \{(\hat{x},\hat{y},\hat{z}) \in \mathbb{C}^3 : 4x^3\hat{x} + 4y^3\hat{y} + 4z^3\hat{z} = 0\}.$$

Explicit bases can be given over the subsets $x \neq 0$, $y \neq 0$, resp. $z \neq 0$.

4. [1]Let $M := \{(x^2, y^2, z^2, xz, yz, xy) \in \mathbb{R}^6 : (x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 = 1\}$.

a) Show that M is a submanifold of $\mathbb{R}^6$. What is the dimension of M?

b) What are the tangent spaces of M?

c) * Prove that M is diffeomorphic to the projective plane $\mathbb{R}P^2 \cong S^2/\{\pm 1\}$.

Hint: Look at the smooth map $f : S^2 \to \mathbb{R}^6 : (x,y,z) \to (x^2, y^2, z^2, xz, yz, xy)$. The image of f is M and f is not injective. In fact, f is quite special and has to do with c).

Solution:

a) We can check that f is a double sheeted covering map and that $f(v) = f(-v)$. Thus take the open set $U_i := \{x \in S^2 : x_i > 0\}$ for $i = 1,2,3$ and notice that $f|_{U_i}$ is injective for $i = 1,2,3$. Furthermore, $M = \cup_{i=1,2,3}(f(U_i) \cap M)$. Thus we write

[1]Exercises with * are reserved for the interested student.

down the charts / parametrisations with $\Omega = B_1(0) \subset \mathbb{R}^2$

$$\begin{aligned}
&\psi_1 : \Omega \to \mathbb{R}^6 : (x_1, x_2) \mapsto f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}),\\
&\varphi_1 = \psi_1^{-1} : M \cap \{y \in \mathbb{R}^6 : y_3 > 0\} \to \Omega : y \mapsto \left(\frac{y_4}{\sqrt{y_3}}, \frac{y_5}{\sqrt{y_3}}\right),\\
&\psi_2 : \Omega \to \mathbb{R}^6 : (x_1, x_3) \mapsto f(x_1, \sqrt{1 - x_1^2 - x_3^2}, x_3),\\
&\varphi_2 = \psi_2^{-1} : M \cap \{y \in \mathbb{R}^6 : y_2 > 0\} \to \Omega : y \mapsto \left(\frac{y_6}{\sqrt{y_2}}, \frac{y_5}{\sqrt{y_2}}\right),\\
&\psi_3 : \Omega \to \mathbb{R}^6 : (x_2, x_3) \mapsto f(\sqrt{1 - x_2^2 - x_3^2}, x_2, x_3),\\
&\varphi_3 = \psi_3^{-1} : M \cap \{y \in \mathbb{R}^6 : y_1 > 0\} \to \Omega : y \mapsto \left(\frac{y_6}{\sqrt{y_1}}, \frac{y_4}{\sqrt{y_1}}\right),
\end{aligned}$$

which form together an atlas for M. Thus M is a 2-dimensional submanifold of $\mathbb{R}^6$.

b) We simply look at the three parts of M parametrised by ψ_i for $i \in 1, 2, 3$. Take for example $y \in M$ with $y_3 > 0$, then

$$T_y M = \text{image}\left(d\psi_1\left(\frac{y_4}{\sqrt{y_3}}, \frac{y_5}{\sqrt{y_3}}\right) : \mathbb{R}^2 \to \mathbb{R}^6\right).$$

As $y \in M$, we have $\sqrt{1 - \left(\frac{y_4}{\sqrt{y_3}}\right)^2 - \left(\frac{y_5}{\sqrt{y_3}}\right)^2} = \sqrt{y_3}$ and

$$\begin{aligned}
\psi_1(x_1, x_2) &= (x_1^2, x_2^2, 1 - x_1^2 - x_2^2, x_1\sqrt{1 - x_1^2 - x_2^2}, x_2\sqrt{1 - x_1^2 - x_2^2}, x_1 x_2),\\
\partial_1\psi_1(x_1, x_2) &= (2x_1, 0, -2x_1, \sqrt{1 - x_1^2 - x_2^2} - \frac{x_1^2}{\sqrt{1 - x_1^2 - x_2^2}}, \frac{-x_1 x_2}{\sqrt{1 - x_1^2 - x_2^2}}, x_2),\\
\partial_2\psi_1(x_1, x_2) &= (0, 2x_2, -2x_2, \frac{-x_1 x_2}{\sqrt{1 - x_1^2 - x_2^2}}, \sqrt{1 - x_1^2 - x_2^2} - \frac{x_2^2}{\sqrt{1 - x_1^2 - x_2^2}}, x_1),\\
v_1(y) &:= \sqrt{y_3}\,\partial_1\psi_1\left(\frac{y_4}{\sqrt{y_3}}, \frac{y_5}{\sqrt{y_3}}\right) = \left(2y_4, 0, -2y_4, y_3 - \frac{y_4^2}{y_3}, -y_4 y_5, y_5\right),\\
v_2(y) &:= \sqrt{y_3}\,\partial_2\psi_1\left(\frac{y_4}{\sqrt{y_3}}, \frac{y_5}{\sqrt{y_3}}\right) = \left(0, 2y_5, -2y_5, -y_4 y_5, y_3 - \frac{y_5^2}{y_3}, y_4\right).
\end{aligned}$$

So finally, we get

$$T_y M = \text{span}\{v_1(y), v_2(y)\}.$$

Bozemoi. The other cases are left to the masochistic reader :)

c) f is a two-sheeted covering map from S^2 to M and $f^{-1}(f(p)) = \{\pm p\}$ for all $p \in S^2$. So $M \cong S^2/\{\pm 1\} \cong \mathbb{R}P^2$ as intrinsic manifolds.

5. Decide for the following maps whether...

i) ...they are injective.

ii) ...they are an immersion.

iii) ...they are proper.

iv) ...their image is a submanifold.

a) $f : S^n \subset \mathbb{R}^{n+1} \to \mathbb{R}^{2n} : x \mapsto (x_1, \ldots, x_n, x_{n+1}x_1, \ldots, x_{n+1}x_n)$.

b) $g : \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right) \to \mathbb{R}^2 : t \mapsto (\cos(t), \cos(t)\sin(t))$.

c) $h : (-\pi, \pi) \to \mathbb{R}^2 : t \mapsto (\cos(t), \sin(t))$.

d) $i : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (\cos(t^5), \sin(t^5))$.

e) $j : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t^2, t^3)$.

f) $k : \mathbb{R} \to \mathbb{R} : t \mapsto t^3$.

Solution:

a)

i) Not injective: North and South pole get mapped to the origin.

ii) Immersion: The differential $df(x) : T_xS^n \to \mathbb{R}^{2n}$ is given by

$$df(x)\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n, x_1\hat{x}_{n+1} + \hat{x}_1 x_{n+1}, \ldots, x_n\hat{x}_{n+1} + \hat{x}_n x_{n+1}).$$

By solving $df(x)\hat{x} = 0$, we see $(\hat{x}_1, \ldots, \hat{x}_n) = 0$. Now if $(x_1, \ldots, x_n) \neq 0$, then also $\hat{x}_{n+1} = 0$. In the opposite case, x is the North or South pole. However, in these points the tangent spaces are given by the equation $\hat{x}_{n+1} = 0$. Thus we have injectivity of $df(x)$ for all $x \in S^n$.

iii) Proper: S^n is compact, so the inverse image of a compact set (closed by Hausdorff condition) is a closed set in a compact space and thus compact.

iv) Not a manifold: Too many tangent vectors around 0. From curves γ around the North pole, we get paths $f \circ \gamma$ which span the space $\operatorname{span}\{e_i + e_{n+i}\}_{i=1,\ldots,n}$ and from curves γ around the Sorth pole, we get paths $f \circ \gamma$ which span the space $\operatorname{span}\{e_i - e_{n+i}\}_{i=1,\ldots,n}$ where e_i is the i-th standard vector in $\mathbb{R}^{2n}$. Thus all curves through the origin of $f(S^n)$ give a tangent space of dimension $2n$. However, $f(S^n) \setminus \{0\}$ is a manifold of dimension n by i), ii), and iii). So it cannot be a manifold.

b)

i) This is a parametrized version of a) in the case $n = 1$: However, $\left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right) \to S^1 \setminus \{(0, -1)\}$, $t \mapsto (\cos(t), \sin(t))$, misses the south pole and g is injective.

ii) Immersion.

iii) Not proper: The compact set $K := \operatorname{image}(\gamma)$ for $\gamma : [\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}^2 : t \mapsto (\cos(t), \cos(t)\sin(t))$ is contained in $\operatorname{image}(g)$, but $f^{-1}(K) = (-\pi, -\frac{3\pi}{2}] \cup \{0\} \cup [\frac{\pi}{2}, \pi)$ which is not compact.

iv) Manifold by a).

c)

i) Injective.

ii) Immersion.

iii) Proper.

iv) Manifold, it's S^1.

d)

i) Not injective.

ii) Not immersion.

iii) Not proper.

iv) Manifold, it's S^1.

e) i) Injective.

ii) Not immersion: $dj(0)\hat{t} = (2t\hat{t}, 3t^2\hat{t})|_{t=0} = 0$.

iii) Proper. j is a homeomorphism.

iv) Not a manifold. It's the cusp $x^3 - y^2 = 0$. It's definitely not a 0- or 2-dimensional manifold by last week's exercise sheet. Thus assume for a contradiction, that $j(\mathbb{R})$ is a manifold around 0. By theorem 1, there is $f : U \to \mathbb{R}$ such that $df(0)$ is surjective and $j(\mathbb{R}) \cap U = f^{-1}(0)$. We see that $\partial_x f(0) = 0$ by implicit derivation and so we need that $\partial_y f(0) \neq 0$. Thus by implicit function theorem, $j(\mathbb{R}) \cap U$ for U maybe smaller, has to be the graph over x. But this is impossible, as there is no $y(x)$ for $x < 0$ and two possibilities for y, namely $\pm\sqrt{x^3}$, for $x > 0$. This is a contradiction.

f) i) Injective.

ii) Not an immersion.

iii) Proper, as it's a homeomorphism.

iv) Manifold. $k(\mathbb{R}) = \mathbb{R}$.

Thus we have established the following table.

	Injective	Immersion	Proper	Manifold
i)	No.	Yes.	Yes.	No.
ii)	Yes.	Yes.	No.	No.
iii)	Yes.	Yes.	Yes.	Yes.
iv)	No.	No.	No.	Yes.
v)	Yes.	No.	Yes.	No.
vi)	Yes.	No.	Yes.	Yes.

6. Recall that we call a map between manifolds $f : M \subset \mathbb{R}^k \to N \subset \mathbb{R}^\ell$ proper, if $f^{-1}(K)$ is compact, whenever $K \subset f(M)$ is compact.

a) Show that the following are equivalent for continuous maps:

- $f^{-1}(K)$ is compact, for all $K \subset N$ compact.
- f is proper and $f(M)$ is closed.

b) Let M be an open subset of a manifold $N \subset \mathbb{R}^k$. Prove that the inclusion map $i : M \to N$ is proper.

Solution:

a) For the first implication, if $K \subset f(M)$ is compact, then it is compact as subset of N as well, so f is proper. For closeness, take $y \in N$ such that there is $\{y_i = f(x_i)\}_{i \in \mathbb{N}}$ converging to y. Then $K = \{y_i, y : i \in \mathbb{N}\}$ is compact in N, so x_i has a convergent subsequence, say to $x \in M$ as $f^{-1}(K)$ is compact. But then we have $f(x_i)$ converges to y and to f(x) by continuity. So $y = f(x)$ and $f(M)$ is closed.

For the converse, any compact set K is closed in a Hausdorff space, so we have that $K \cap f(M)$ is closed and any closed subset of a compact set is compact again. Therefore, $f^{-1}(K) = f^{-1}(K \cap f(M))$ is compact, by properness of f.

b) Sort of a tautology. A compact subset contained in M is a compact subset of M.