Solution 3

1. Calculate the flow of the following vector fields and find all their zeros.

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$

c) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, -y)$

d) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y, -x)$

e) $f : S^2 \to \mathbb{R}^3$, $f(p) = e_3 \times p$ with $e_3 = (0, 0, 1)$

f) $f : S^2 \to \mathbb{R}^3$, $f(p) = (e_3 \times p) \times p$ with $e_3 = (0, 0, 1)$

g) $f : S^3 \to \mathbb{R}^4$, $f(x_1, y_1, x_2, y_2) = (-y_1, x_1, -y_2, x_2)$

Solution:

a) We need to solve the differential equation $\dot{x}(t) = x(t)^2$, $x(0) = x_0$. The only zero of f is $x_0 = 0$ and in this case the solution is constant. If $x_0 \neq 0$ separation of variables yields

$$\int_{x_0}^{x(t)} \frac{d\xi}{\xi^2} = \int_0^t ds.$$

This gives $1/x_0 - 1/x(t) = t$ and hence

$$x(t) = \frac{x_0}{1 - tx_0}.$$

The maximal existence intervall of this solution is $I(x_0) = \{t \in \mathbb{R} \,|tx_0 < 1\}$ and the flow is given by

$$\phi : \{(t, x) \in \mathbb{R} \times \mathbb{R} \,|\, tx < 1\} \to \mathbb{R}, \qquad \phi(t, x) = \frac{x}{1 - tx}$$

b) We need to solve the differential equation $\dot{x}(t) = x(t)^2 - 1$, $x(0) = x_0$. The zeros of f are $x_0 = \pm 1$ and for these initial values the solution is constant. Otherwise separation of variables yields

$$\int_{x_0}^{x(t)} \frac{d\xi}{\xi^2 - 1} = \int_0^t ds$$

The left hand side integrates to

$$\int \frac{d\xi}{\xi^2 - 1} = \frac{-1}{2} \int \frac{1}{\xi + 1} - \frac{1}{\xi - 1} d\xi = -\frac{1}{2} \log \left| \frac{1 + \xi}{1 - \xi} \right|.$$

For $\xi \in (-1, 1)$ the right hand side agrees with $-\operatorname{arctanh}(\xi)$. In this case the solution exists for all $t \in \mathbb{R}$ and it is given by

$$x(t) = \tanh\left(\operatorname{arctanh}(x_0) - t\right).$$

For $x_0 > 1$ the solution exists for $t \in I(x_0) = (-\infty, \frac{1}{2} \log(\frac{1 - x_0}{1 + x_0}))$ and it is given by

$$x(t) = \frac{-(x_0 - 1) - (1 + x_0)e^{-2t}}{(x_0 - 1) - (1 + x_0)e^{-2t}}.$$

For $x_0 < -1$ the solution exists for $t \in I(x_0) = (\frac{1}{2}\log(\frac{x_0+1}{x_0-1}), \infty)$ and it is given by

$$x(t) = \frac{x_0 - 1 + (1+x_0)e^{-2t}}{1 - x_0 + (1+x_0)e^{-2t}}.$$

c) The only zero of the vector field is $(x_0, y_0) = (0, 0)$. We need to solve the differential equation

$$(\dot{x}(t), \dot{y}(t)) = (x(t), -y(t)), \qquad (x(0), y(0)) = (x_0, y_0)$$

The solution exists for all time $t \in \mathbb{R}$ and is given by

$$(x(t), y(t)) = (x_0 e^t, y_0 e^{-t}).$$

Hence the flow is given by

$$\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \phi(t, x, y) = (xe^t, ye^{-t}).$$

d) The only zero of the vector field is $(x_0, y_0) = (0, 0)$. We need to solve the differential equation

$$(\dot{x}(t), \dot{y}(t)) = (y(t), -x(t)), \qquad (x(0), y(0)) = (x_0, y_0).$$

This can be solved using the general theory of linear ODEs. Alternatively, note that the equation implies $\ddot{x}(t) = -x(t)$ and the general solution of this equation is

$$x(t) = c_1 \cos(t) + c_2 \sin(t)$$

and thus

$$y(t) = \dot{x}(t) = -c_1 \sin(t) + c_2 \cos(t).$$

The initial condition $x(0) = x_0$ and $y(0) = y_0$ imply $c_1 = x_0$ and $c_2 = y_0$. Hence the solution exists for all $t \in \mathbb{R}$ and the flow is given by

$$\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad \phi(t, x, y) = (\cos(t)x + \sin(t)y, -\sin(t)x + \cos(t)y)$$

e) For $p = (p_1, p_2, p_3) \in S^2 \subset \mathbb{R}^3$ we have

$$f(p) = e_3 \times p = \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix}.$$

In particular, the only zeros of f are the two poles $(0, 0, \pm 1)$. We need to solve the equation

$$\dot{p}(t) = \begin{pmatrix} p_2(t) \\ -p_1(t) \\ 0 \end{pmatrix}.$$

The solution of this equation follows from the previous part and we obtain:

$$\phi : \mathbb{R} \times S^2 \to S^2, \qquad \phi(t, p) = \begin{pmatrix} \cos(t)p_1 - \sin(t)p_2 \\ \sin(t)p_1 + \cos(t)p_2 \\ p_3 \end{pmatrix}$$

f) For $p = (p_1, p_2, p_3) \in S^2 \subset \mathbb{R}^3$ we have

$$f(p) = (e_3 \times p) \times p = \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1 p_3 \\ p_2 p_3 \\ -p_1^2 - p_2^2 \end{pmatrix}.$$

In particular, the only zeros of f are the two poles $(0, 0, \pm 1)$. Using the equation $-p_1^2 - p_2^2 = p_3^2 - 1$ we need to solve the equation

$$\dot{p}(t) = \begin{pmatrix} p_1(t)p_3(t) \\ p_2(t)p_3(t) \\ p_3(t)^2 - 1 \end{pmatrix}, \qquad p(0) = p^0 = \begin{pmatrix} p_1^0 \\ p_2^0 \\ p_3^0 \end{pmatrix}.$$

Suppose $p^0 \neq (0, 0, \pm 1)$. Then $p_3^0 \in (-1, 1)$ and it follows from our calculation in part (b) that

$$p_3(t) = \tanh\Big(\operatorname{arctanh}(p_3^0) - t\Big).$$

The equations for $p_1(t)$ and $p_2(t)$ are linear first order equations, which can be solved using separation of varables:

$$p_1(t) = p_1^0 \exp\left(\int_0^t p_3(s)\, ds\right), \qquad p_2(t) = p_2^0 \exp\left(\int_0^t p_3(s)\, ds\right)$$

Using $\int \tanh(x)\, dx = \log(\cosh(x))$, we have

$$\exp\left(\int_0^t p_3(s)\, ds\right) = \frac{\cosh(\operatorname{arctanh}(p_3^0))}{\cosh\left(\operatorname{arctanh}(p_3^0) - t\right)}$$

and the expressions above simplify to

$$p_i(t) = \frac{p_i^0 \cosh(\operatorname{arctanh}(p_3^0))}{\cosh\left(\operatorname{arctanh}(p_3^0) - t\right)}$$

for $i = 1, 2$. Therefore, the flow $\phi : \mathbb{R} \times S^2 \to S^2$ is given by $\phi(t, p) = p$ for $p = (0, 0, \pm 1)$ and

$$\phi(t, p) = \begin{pmatrix} p_1 \cosh(\operatorname{arctanh}(p_3)) / \cosh(\operatorname{arctanh}(p_3) - t) \\ p_2 \cosh(\operatorname{arctanh}(p_3)) / \cosh(\operatorname{arctanh}(p_3) - t) \\ \tanh(\operatorname{arctanh}(p_3) - t) \end{pmatrix}$$

for $p \in S^2 \backslash \{(0, 0, \pm 1)\}$.

g) The vector field has no zeros and we need to solve the equation

$$(\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2) = (-y_1, x_1, -y_2, x_2), \qquad (x_1(0), y_1(0), x_2(0), y_2(0)) = (x_1^0, y_1^0, x_2^0, y_2^0).$$

This equation uncouples and the pairs (x_1, y_1) and (x_2, y_2) satisfy the equation form part (d) respectively. Hence we get for the flow

$$\phi : \mathbb{R} \times S^3 \to S^3, \qquad \phi(t, x_1, y_1, x_2, y_2) = \begin{pmatrix} \cos(t)x_1 + \sin(t)y_1 \\ -\sin(t)x_1 + \cos(t)y_1 \\ \cos(t)x_2 + \sin(t)y_2 \\ -\sin(t)x_2 + \cos(t)y_2 \end{pmatrix}$$

2. The goal of this exercise is to prove the following.

$$\text{Every vector field on a compact manifold is complete.} \tag{1}$$

Thus, let $M \subset \mathbb{R}^k$ be a compact manifold and let $X : M \to \mathbb{R}^k$ be a vector field on M. We need to prove the following steps.

a) Use Theorem 8 to prove that $U_\epsilon := \{p \in M : [-\epsilon, \epsilon] \subset I(p)\}$ is open for all $\epsilon > 0$.

b) Use Theorem 7 to prove that there is $\epsilon_0 > 0$ such that $M = U_{\epsilon_0}$.

c) Use a) and b) to prove (1).

Solution:

a) Fix $\epsilon > 0$ and take $p \in U_\epsilon$. Thus $(\epsilon, p) \in \mathcal{D}$ which is open by Theorem 8. So there is a (basic) product open set $J \times V_+ \subset \mathcal{D}$ such that $(\epsilon, p) \in J \times V_+$. This means that $\epsilon \in I(q)$ for all $q \in V_+$. Similarly, there is an open set V_- such that $p \in V_-$ and $-\epsilon \in I(q)$ for all $q \in V_-$. Thus $V_- \cap V+ \subset U_\epsilon$ is an open set around p. As p was arbitrary, U_ϵ is open.

b) By Theorem 7, for every $p \in M$, there is $\epsilon > 0$ such that $[-\epsilon, \epsilon] \subset I(p)$. In other words, $M = \bigcup_{\epsilon > 0} U_\epsilon$ is a nested sequence ($U_\epsilon \subset U_{\epsilon'}$ when $\epsilon' < \epsilon$) of open sets covering the compact set M. So there is a finite subcover or in other words, there must be ϵ_0 such that $U_{\epsilon_0} = M$.

c) Let ϵ_0 be as in b). We use the formula $I(p) = I(\phi(s, p)) + s$ of Theorem 8. Pick $p_0 \in M$ and $t \in I(p)$, $t > 0$. Take an integer $k \in \mathbb{Z}$ such that $k\epsilon_0 > t$. Then we iterate

$$I(p) = I(\phi(\epsilon_0, p)) + \epsilon_0 = I(\phi(\epsilon_0, \phi(\epsilon_0, p))) + 2\epsilon_0 = \ldots = I(\phi(\epsilon_0, \ldots, \phi(\epsilon_0, p))) + k\epsilon_0.$$

As $0 \in I(q)$ for all $q \in M$, $k\epsilon_0 \in I(p)$. Therefore, $t \in [0, k\epsilon_0] \subset I(p)$ and the same argument can be repeated for any $t < 0$. Thus, $I(p) = \mathbb{R}$ and as $p \in M$ was arbitrary, X is complete.

3. Consider the vector fields X, Y, Z on S^2 given by

$$X(p) = \xi \times p, \quad Y(p) = \eta \times p, \quad Z(p) = (\xi \times p) \times p,$$

for $\xi, \eta \in S^2$ and $\xi \neq \eta$. Calculate the Lie brackets $[X, Y]$, $[X, Z]$ and $[Y, Z]$.

Hint: Take $\xi = e_3$ to ease the calculations.

Solution: It is enough to take $\xi = e_3$ (Simply rotate the sphere by an element of $SO(3)$ sending ξ to e_3 and use Exercise 4 c)). Thus from exercise 1, we already have the expressions

$$X(p) = \begin{pmatrix} -p_2 \\ p_1 \\ 0 \end{pmatrix}, \quad Z(p) = \begin{pmatrix} p_1 p_3 \\ p_2 p_3 \\ -1 + p_3^2 \end{pmatrix}.$$

The last one is $Y(p) = \begin{pmatrix} \eta_2 p_3 - \eta_3 p_2 \\ \eta_3 p_1 - \eta_1 p_3 \\ \eta_1 p_2 - \eta_2 p_1 \end{pmatrix}$. We also calculate differentials as

$$dX(p)\hat{p} = \begin{pmatrix} -\hat{p}_2 \\ \hat{p}_1 \\ 0 \end{pmatrix}, \quad dY(p)\hat{p} = \begin{pmatrix} \eta_2 \hat{p}_3 - \eta_3 \hat{p}_2 \\ \eta_3 \hat{p}_1 - \eta_1 \hat{p}_3 \\ \eta_1 \hat{p}_2 - \eta_2 \hat{p}_1 \end{pmatrix} \quad dZ(p)\hat{p} = \begin{pmatrix} \hat{p}_1 p_3 + p_1 \hat{p}_3 \\ \hat{p}_2 p_3 + p_2 \hat{p}_3 \\ 2 p_3 \hat{p}_3 \end{pmatrix}.$$

So we plug in the formula for the Lie bracket.

$$[X,Y](p) = dX(p)Y(p) - dY(p)X(p) = \begin{pmatrix} \eta_1 p_3 \\ \eta_2 p_3 \\ -\eta_1 p_1 - \eta_2 p_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = (\xi \times \eta) \times p,$$

$$[Y,Z](p) = dY(p)Z(p) - dZ(p)Y(p) = \begin{pmatrix} -\eta_2 - \eta_1 p_1 p_2 + \eta_2 p_1^2 \\ \eta_1 - \eta_1 p_2^2 + \eta_2 p_1 p_2 \\ -\eta_1 p_2 p_3 + \eta_2 p_1 p_3 \end{pmatrix} = p \times (p \times (\eta \times \xi))$$

$$[X,Z](p) = dX(p)Z(p) - dZ(p)X(p) = \begin{pmatrix} -p_2 p_3 \\ p_1 p_3 \\ 0 \end{pmatrix} - \begin{pmatrix} -p_2 p_3 \\ p_1 p_3 \\ 0 \end{pmatrix} = 0$$

4. Assume X, Y, Z are complete vector fields on M.

a) Let φ, ψ be diffeomorphisms on M. Prove that $\varphi^*\psi^*X = (\psi \circ \varphi)^*X$.

b) Let φ be a diffeomorphism on M and let ψ^t be the flow of Y for $t \in \mathbb{R}$. Prove that $\varphi^{-1} \circ \psi^t \circ \varphi$ is the flow of φ^*Y for $t \in \mathbb{R}$.

c) Let φ be a diffeomorphism on M. Prove that $\varphi^*[X,Y] = [\varphi^*X, \varphi^*Y]$.

d) Prove the Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Hint: For c), use the formula $[X,Y] = \frac{d}{dt}\big|_{t=0} (\psi^t)^*X$ where ψ^t if the flow of Y together with a) and b). For d), look at c) with φ replaced by φ^t, where φ^t is the flow of Z and differentiate with respect to $t \in \mathbb{R}$. Recall that the Lie bracket is a bi-linear map.

Solution:

a) By Definition $\psi^*X(p) = d\psi(p)^{-1}X(\psi(p))$. Using the chain rule we obtain

$$\begin{aligned} \varphi^*\psi^*X(p) &= d\varphi(p)^{-1}(\psi^*X)(\varphi(p)) \\ &= d\varphi(p)^{-1} d\psi(\varphi(p))^{-1} X(\psi(\varphi(p))) \\ &= d(\psi \circ \varphi)(p)^{-1} X((\psi \circ \varphi)(p) \\ &= (\psi \circ \varphi)^* X(p) \end{aligned}$$

and this proves the claim.

b) Using the chain rule and the fact that ψ^t is the flow of Y, we obtain:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_0} (\varphi^{-1} \circ \psi^t \circ \varphi)(p) &= d\varphi^{-1}((\psi^{t_0} \circ \varphi)(p)) \left(\frac{d}{dt}\bigg|_{t=t_0} \psi^t(\varphi(p)) \right) \\ &= d\varphi((\varphi^{-1} \circ \psi^{t_0} \circ \varphi)(p))^{-1} Y((\psi^{t_0} \circ \varphi)(p)) \\ &= (\varphi^*Y)((\varphi^{-1} \circ \psi^{t_0} \circ \varphi)(p)) \end{aligned}$$

Hence $\varphi^{-1} \circ \psi^t \circ \varphi$ is the flow of φ^*Y.

c) We use the formula

$$[X,Y] = \frac{d}{dt}\bigg|_{t=0} (\psi^t)^*X$$

where ψ^t denotes the flow of Y. Together with the fromulae from part a) and b) we obtain:

$$\varphi^*[X,Y] = \frac{d}{dt}\bigg|_{t=0} \varphi^*(\psi^t)^*X = \frac{d}{dt}\bigg|_{t=0} (\varphi^{-1} \circ \psi^t \circ \varphi)^* \varphi^* X = [\varphi^*X, \varphi^*Y]$$

d) Denote the flow of Z by φ^t. From part c) follows $(\varphi^t)^*[X,Y] = [(\varphi^t)^*X, (\varphi^t)^*Y]$ for all $t \in \mathbb{R}$. Hence

$$\begin{aligned}[Z,[X,Y]] &= \frac{d}{dt}\bigg|_{t=0} (\varphi^t)^*[X,Y] = \frac{d}{dt}\bigg|_{t=0} [(\varphi^t)^*X, (\varphi^t)^*Y] \\ &= \left[\frac{d}{dt}\bigg|_{t=0} (\varphi^t)^*X, Y\right] + \left[X, \frac{d}{dt}\bigg|_{t=0} (\varphi^t)^*Y\right] \\ &= [[Z,X],Y] + [X,[Z,Y]].\end{aligned}$$

This is equivalent to the Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

5. A car is parking on the side of the street. Its present position is given by its position $(x_1(t), x_2(t)) \in \mathbb{R}^2$ and the direction $\theta(t)$. (Here $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the angle of the axis of the car with the x_1 coordinate axis.) We assume that the velocity of the car always points in the direction of the car's axis. We consider the vector fields $X(x_1, x_2, \theta) := (\cos(\theta), \sin(\theta), 1)$, $Y(x_1, x_2, \theta) := (\cos(\theta), \sin(\theta), -1)$ on the phase space $P := \mathbb{R}^2 \times S^1$.

a) Show that X, Y are complete and calculate their flows $\varphi^t, \psi^t : P \to P$ for $t \in \mathbb{R}$. Also consider the curve $\chi : \mathbb{R} \to \mathrm{Diff}(P)$ given by

$$\chi(t) := \varphi^t \circ \psi^t \circ \varphi^{-t} \circ \psi^{-t}.$$

Calculate $\dot{\chi}(t)$ for $t \in \mathbb{R}$ and $1/2\,\ddot{\chi}(0)$.

b) Calculate the vector field $[X,Y]$ and the flow of $[X,Y]$.

Solution:

a) We calculate the flow of X. For this we need to solve the equation

$$(\dot{x}_1(t), \dot{x}_2(t), \dot{\theta}(t)) = (\cos(\theta(t)), \sin(\theta(t)), 1)$$

For given initial conditions $(x_1(0), x_2(0), \theta(0)) = (x_1^0, x_2^0, \theta^0)$. The equation for θ has the solution $\theta(t) = t + \theta^0$. This yields in turn:

$$x_1(t) = x_1^0 + \int_0^s \cos(\theta(s))\, ds = x_1^0 + \sin(\theta^0 + t) - \sin(\theta^0)$$

$$x_2(t) = x_2^0 + \int_0^s \sin(\theta(s))\, ds = x_1^0 - \cos(\theta^0 + t) + \cos(\theta^0)$$

Therefore, the flow of X is given by the equation

$$\varphi(t, x_1, x_2, \theta) = (x_1 + \sin(\theta + t) - \sin(\theta), x_2 - \cos(\theta + t) + \cos(\theta), \theta + t)$$

A similar calculation yields for the flow of Y the formula

$$\psi(t, x_1, x_2, \theta) = (x_1 - \sin(\theta - t) + \sin(\theta), x_2 + \cos(\theta - t) - \cos(\theta), \theta - t)$$

For the following calculation, it is usefull to identify $(x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + \mathbf{i}x_2 \in \mathbb{C}$. In this notation the flows are given by:

$$\begin{aligned}\varphi^t(z,\theta) &= \left(z + \mathbf{i}e^{\mathbf{i}\theta} - \mathbf{i}e^{\mathbf{i}(t+\theta)}, \theta + t\right) \\ \psi^t(z,\theta) &= \left(z - \mathbf{i}e^{\mathbf{i}\theta} + \mathbf{i}e^{\mathbf{i}(\theta-t)}, \theta - t\right).\end{aligned}$$

Then follows

$$\chi(t)(z,\theta) = (\varphi^t \circ \psi^t \circ \varphi^{-t} \circ \psi^{-t})(z,\theta) = (z + 2\mathbf{i}e^{\mathbf{i}(\theta+t)} + 2\mathbf{i}e^{\mathbf{i}(\theta-t)} - 4\mathbf{i}e^{\mathbf{i}\theta}, \theta)$$

and finally

$$\dot{\chi}(t)(z,\theta) = (-2e^{\mathbf{i}(\theta+t)} - 2\mathbf{i}e^{\mathbf{i}(\theta-t)}, 0), \qquad \frac{1}{2}\ddot{\chi}(0)(z,\theta) = (-2\mathbf{i}e^{\mathbf{i}\theta}, 0).$$

b) We have

$$dX(x,\theta)[\hat{x},\hat{\theta}] = (-\sin(\theta)\hat{\theta}, \cos(\theta)\hat{\theta}, 0)$$
$$dY(x,\theta)[\hat{x},\hat{\theta}] = (-\sin(\theta)\hat{\theta}, \cos(\theta)\hat{\theta}, 0)$$

and hence

$$[X,Y](x,\theta) = dX(x,\theta)Y(x,\theta) - dY(x,\theta)X(x,\theta) = 2(\sin(\theta), -\cos(\theta), 0).$$

We observe that $[X,Y] = \frac{1}{2}\ddot{\chi}(0)$. This a general fact, established in Lemma 2.4.18. The flow of $[X,Y]$ is given by the map

$$\phi : \mathbb{R} \times P \to P, \qquad \phi(t,x,\theta) = (x,\theta) + 2t(\sin(\theta), -\cos(\theta), 0).$$

The geometric significance of this calculation is explained in Example 2.4.25 in the lecture notes as follows: The vector field $[X,Y]$ represents a sideways move of the car to the right. And a sideways move by $2\epsilon^2$ can be achieved by following a backward right turn for time ϵ, then a backward left turn for time ϵ, then a forward right turn for time ϵ, and finally a forward left turn for time ϵ.

6. Let $\mathbb{H} \cong \mathbb{R}^4$ be the quaternions, $Sp(1) = S^3$ be the group of unit quaternions and $\operatorname{Im}\mathbb{H} := \{x_1 i + x_2 j + x_3 k : (x_1,x_2,x_3) \in \mathbb{R}^3\} \cong \mathbb{R}^3$ be the imaginary quaternions. Recall that $\bar{x} = x_0 - x_1 i - x_2 j - x_3 k$ and $x^{-1} = \frac{\bar{x}}{|x|^2}$.

a) Let $\Phi : Sp(1) \to SO(3)$ be the map $x \mapsto \Phi(x)$ with

$$\Phi(x) : \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3 \to \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3 \text{ is given by } \Phi(x)\xi = x\xi\bar{x}.$$

Write $\Phi(x)$ as a matrix and prove that this is a group homomorphism and a smooth double cover. Compute $d\Phi(1) : \mathfrak{sp}(1) = \operatorname{Im}\mathbb{H} \mapsto \mathfrak{so}(3)$ and conclude that this is a Lie algebra isomorphism. What is the Lie bracket on $\mathbb{R}^3 \cong \operatorname{Im}(\mathbb{H})$ you get?

b) Let $\Psi : Sp(1) \to SU(2)$ be the map $x \mapsto \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$. Prove that this is a group isomorphism and compute $d\Psi(1) : \mathfrak{sp}(1) \to \mathfrak{su}(2)$.

Thus we proved one of the so called accidental isomorphism

$$\mathfrak{sp}(1) \cong \mathfrak{su}(2) \cong \mathfrak{so}(3).$$

Solution:

a) For $x \in Sp(1)$, one has $\bar{x} = x^{-1}$ and conjugation with x yields the group homomorphism $c_x : Sp(1) \to Sp(1)$, $c_x(z) := xz\bar{x}$. We observe $\operatorname{Im}(\mathbb{H}) = T_1 Sp(1)$ and hence

$$\Phi(x) = dc_x(1) : \operatorname{Im}(\mathbb{H}) \to \operatorname{Im}(\mathbb{H})$$

is well-defined.

It is clearly a group homomorphism: For $x,y \in Sp(1)$ and $\xi \in \operatorname{Im}(\mathbb{H})$ it holds

$$\Phi(xy)\xi = xy\xi\bar{y}\bar{x} = x(y\xi\bar{y})\bar{x} = \Phi(x)\Phi(y)\xi.$$

and hence $\Phi(xy) = \Phi(x)\Phi(y)$.

The kernel of Φ is given by $\{\pm 1\}$. Indeed, $x \in \ker(\Phi)$ if and only if

$$\forall z \in \operatorname{Im}(\mathbb{H} : \ xz\bar{x} = z \quad \iff \quad \forall z \in \operatorname{Im}(\mathbb{H} : \ xz = zx$$

where we useed $|x|^2 = \bar{x}x = 1$. The later condition is equivalent to $x \in \mathbb{R}$ and thus $\ker(\Phi) = Sp(1) \cap \mathbb{R} = \{\pm 1\}$.

We show $\Phi(x) \in \mathrm{SO}(3)$. The standard inner product on $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ can be expressed by $\langle \xi, \eta \rangle = \mathrm{Re}(\bar{\xi}\eta)$. Then

$$\langle \Phi(x)\xi, \Phi(x)\eta \rangle = \langle x\xi\bar{x}, x\eta\bar{x} \rangle = \mathrm{Re}(x(\bar{\xi}\eta)\bar{x})$$

Decompose $\bar{\xi}\eta = \mathrm{Re}(\bar{\xi}\eta) + \zeta$ with $\zeta \in \mathrm{Im}(\mathbb{H})$. We have seen above that $x\zeta\bar{x} = \Phi(x)\zeta \in \mathrm{Im}(\mathbb{H})$ and hence

$$\mathrm{Re}(x(\bar{\xi}\eta)\bar{x}) = \mathrm{Re}(x\mathrm{Re}(\bar{\xi}\eta)\bar{x}) = \mathrm{Re}(\bar{\xi}\eta) = \langle \xi, \eta \rangle.$$

This shows $\Phi(x) \in O(3)$. Since the image of Φ is connected and containes $\mathrm{Id}_{\mathrm{Im}(\mathbb{H})} = \Phi(1)$, it is moreover contained in $\mathrm{SO}(3)$.

We show that $\Phi : S^3 \to \mathrm{SO}(3)$ is surjective. For this we use that the derivative

$$d\Phi(1) : \mathrm{Im}(\mathbb{H}) \to T_1\mathrm{SO}(3) =: \mathfrak{so}(3)$$

is bijective, which we verify below. It then follows from the implicit function theorem that Φ is a local diffeomorphism around $1 \in S^3$ onto its image. In particular, the image of Φ contains an open neighborhood $\mathrm{Id}_{\mathrm{Im}(\mathbb{H})} \in U \subset \mathrm{SO}(3)$ of the identity. For any point $\Phi(x)$ in the image, $U\Phi(x) = \{\Psi\Phi(x) \mid \Psi \in U\}$ is an open neighborhood of $\Phi(x)$ contained in the image of Φ. This shows that the image of Φ is an open subset of $\mathrm{SO}(3)$. Since S^3 is compact, the image is also compact and hence closed. Since $\mathrm{SO}(3)$ is connected, it follows that Φ must be surjective.

Finally, we provide some explicit formulae. Write $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, then

$$\Phi(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}$$

The derivative $d\Phi(1)\xi \in \mathfrak{so}(3)$ is the linear map $\eta \mapsto \xi\eta + \eta\bar{\xi}$. This is represented by the following matrix:

$$d\Phi(1)\xi = 2\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Now recall that $\mathfrak{so}(3)$ is the space of skew-symmetric 3×3 matrices. Hence the formula implies directly that $d\Phi(1) : \mathrm{Im}(\mathbb{H}) \to \mathfrak{so}(3)$ is an isomorphism.

The Lie-bracket of $\mathfrak{so}(3)$ is the commutator $[A, B] = AB - BA$. We calculate the induced Lie bracket on $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$. For this it is usefull to express the linear map $d\Phi(1)\xi$ using the cross-product on $\mathbb{R}^3$. One has

$$(d\Phi(1)\xi)v = 2\xi \times v$$

and hence

$$\begin{aligned} [(d\Phi(1)\xi), (d\Phi(1)\eta)]v &= 4\xi \times (\eta \times v) - 4\eta \times (\xi \times v) \\ &= 4\xi \times (\eta \times v) + 4\eta \times (v \times \xi) \\ &= 4(\xi \times \eta) \times v \\ &= d\phi(1)(2\xi \times \eta)v. \end{aligned}$$

We used in the third equation that the cross-product satisfies the Jacobi identity. Hence the induced Lie bracked is given by $[\xi, \eta] = 2\xi \times \eta$.

b) Let $\begin{pmatrix} z & u \\ w & v \end{pmatrix} \in \mathrm{SU}(2)$. For an unitary matrix, the column vector $(z,w)^t$ and $(u,v)^t$ are both of unit lengths and orthogonal:

$$|z|^2 = |w|^2 = 1, \qquad |u|^2 + |v|^2 = 1, \qquad \bar{z}u + \bar{w}v = 0.$$

For fixed (z,w) the last equation has a complex onedimensional space of solutions and hence $(u,v) = \lambda(-\bar{w}, z)$ for some $\lambda \in \mathbb{C}$. Then

$$\det \begin{pmatrix} z & -\lambda\bar{w} \\ w & \lambda z \end{pmatrix} = \lambda(|z|^2 + |w|^2) = \lambda$$

shows that SU(2) consists of those matrices with $\lambda = 1$, i.e.

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & z \end{pmatrix} \in \mathbb{C}^{2\times 2} \,:\, |z|^2 + |w|^2 = 1 \right\}$$

From this description it is clear that $\Psi : Sp(1) \to SU(2)$ is bijective. Its differential is given by

$$d\Psi(1) : \mathrm{Im}(\mathbb{H}) \to \mathfrak{su}(2), \qquad d\Psi(1)\xi = \begin{pmatrix} \mathbf{i}\xi_1 & \xi_2 + \mathbf{i}\xi_3 \\ -\xi_2 + \mathbf{i}\xi_3 & -\mathbf{i}\xi_1 \end{pmatrix}.$$

Recall that $\mathfrak{su}(2) = \{A \in \mathbb{C}^{2\times 2} \,|\, A^* = -A, \, \mathrm{tr}(A) = 0\}$ is the space of traceless skew-hermitian 2×2 matrices. In particular, $d\Psi(1)$ is an isomorphism.

It remains to verify that Ψ is a group homomorphism: For this denote

$$I := \Phi(\mathbf{i}) = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \qquad J := \Phi(\mathbf{j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K := \Phi(\mathbf{k}) = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

Then Ψ has the explicit description

$$\Psi(x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3) = x_0\mathbb{1} + x_1 I + x_2 J + x_3 J.$$

and it is easy to verify that the matrices I, J, K satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = -\mathbb{1}, \qquad IJ = K = -JI.$$

Hence quaternionic multiplication in $Sp(1)$ corresponds to matrix multiplication in $\mathbb{C}^{2\times 2}$ and Ψ is a group homomorphism.