Solution 4

1. Let $f : N \to M$ be a smooth map between two manifolds. Show that the following subsets of N are open:

$$U := \{p \in N \,:\, df(p) : T_pN \to T_{f(p)}M \text{ is surjective}\},$$
$$V := \{p \in N \,:\, df(p) : T_pN \to T_{f(p)}M \text{ is injective}\}.$$

Hint: Prove first that the sets $\{A \in \mathbb{R}^{m\times n} \,:\, \mathrm{rk}(A) = m\}$ and $\{A \in \mathbb{R}^{m\times n} \,:\, \ker(A) = 0\}$ are open subsets of $\mathbb{R}^{m\times n}$. Use this to prove the result in local coordinates.

Solution: We show that $U_0 := \{A \in \mathbb{R}^{m\times n} \,:\, \mathrm{rk}(A) = m\}$ is open. For $m > n$ we have $U_0 = \emptyset$ and the claim is trivial. Hence assume $m \le n$ and let $A \in U_0$ be given. Write $A = (a_1|a_2|\cdots|a_n)$ where $a_i \in \mathbb{R}^m$ are the column vectors of A. Since A has rank m, there exists indices $i_1 < \ldots < i_m$ such that $\tilde{A} = (a_{i_1}|a_{i_2}|\cdots|a_{i_m}) \in \mathbb{R}^{m\times m}$ is invertible. By continuity of the map

$$\Pi : \mathbb{R}^{n\times m} \to \mathbb{R}^{m\times m}, \qquad B = (b_1|b_2|\cdots|b_n) \mapsto \Pi(B) = (b_{i_1}|b_{i_2}|\cdots|b_{i_m})$$

the preimage $\Pi^{-1}(\mathrm{GL}(m,\mathbb{R}))$ in an open neighborhood of A in U_0.

We show that $V_0 := \{A \in \mathbb{R}^{m\times n} \,:\, \ker(A) = 0\}$ is open. Let $A \in V_0$ be given and define $\epsilon := \inf\{||Av|| \,:\, v \in \mathbb{R}^m, ||v|| = 1\}$. Since the unit sphere $S^{m-1} = \{v \in \mathbb{R}^m \,:\, ||v|| = 1\} \subset \mathbb{R}^m$ is compact, the infimum is attained and $\epsilon > 0$. Let $B \in V_0$ be given with $||B - A|| < \epsilon/2$, and $v \in \mathbb{R}^m$ with $||v|| = 1$. Then

$$||Bv|| \ge ||Av|| - ||(B - A)v|| \ge \epsilon - ||B - A|| \ge \epsilon/2.$$

Since v was arbitrary, we have $B \in V_0$ and hence $B_{\epsilon/2}(A) \subset V_0$.

We prove the general statement. Let $p \in U \subset N$ (resp. $p \in V \subset N$) be given and choose charts

$$\phi : \Omega_0 \subset \mathbb{R}^n \to W_0 \subset N, \qquad \psi : \Omega_1 \subset \mathbb{R}^m \to W_1 \subset M$$

where $p \in W_0 \subset N$ and $f(p) \in W_1 \subset M$ are open neighborhoods around p and $f(p)$ such that $W_0 \subset f^{-1}(W_1)$. Define $\tilde{f} : \Omega_0 \to \Omega_1$ by $\tilde{f}(x) := (\psi^{-1} \circ f \circ \phi)(x)$. Then

$$d\tilde{f}(x) = d\psi^{-1}(f(\phi(x))) \circ df(\phi(x)) \circ d\phi(x).$$

Since the differentials $d\phi(x) : \mathbb{R}^n \to T_{\phi(x)}N$ and $d\psi^{-1}(f(\phi(x))) : T_{f(\phi(x))}M \to \mathbb{R}^m$ are both bijective, the derivative $df(\phi(x))$ is surjective (resp. injective) if and only if $d\tilde{f}(x)$ is surjective (resp. injective).

Represent $d\tilde{f}(x) \in \mathbb{R}^{n\times m}$ by its Jacobi matrix. Since f is smooth, the derivative defines a continous map

$$d\tilde{f} : W_0 \to \mathbb{R}^{n\times m}, \qquad x \mapsto d\tilde{f}(x)$$

and it follows that

$$U_1 := \{x \in \Omega_0 \,:\, d\tilde{f}(x) \in U_0\} \quad (\text{resp. } V_1 := \{x \in \Omega_0 \,:\, d\tilde{f}(x) \in V_0\})$$

is an open subset of Ω_0. Therefore $\phi^{-1}(U_1) \subset U$ (resp. $\phi^{-1}(V_1) \subset V$) is an open neighborhood of $p \in U$ (resp. $p \in V$).

2. Let M be a manifold and denote by $\mathcal{F}(M) = C^\infty(M, \mathbb{R})$ the space of smooth functions on M. Recall that every vector field $X \in \mathrm{Vect}(M)$ defines a derivation by the following formula

$$\mathcal{L}_X : \mathcal{F}(M) \to \mathcal{F}(M), \qquad (\mathcal{L}_X f)(p) := df(p)X(p).$$

Verify the identity

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_X \mathcal{L}_Y = -[\mathcal{L}_X, \mathcal{L}_Y],$$

(where the last equation holds by definition).

Solution: Let $f \in \mathcal{F}(M)$ we have $(\mathcal{L}_X f)(p) = df(p)X(p)$. Differentiating the right hand side in direction $Y(p)$ using the chain rule, we obtain

$$(\mathcal{L}_Y \mathcal{L}_X f)(p) = d^2 f(p)(Y(p), X(p)) + df(p)[dX(p)Y(p)].$$

In the same way, one obtains

$$(\mathcal{L}_X \mathcal{L}_Y f)(p) = d^2 f(p)(X(p), Y(p)) + df(p)[dY(p)X(p)].$$

By symmetry of the second derivative it holds $d^2 f(p)(X(p), Y(p)) = d^2 f(p)(Y(p), X(p))$ and therefore

$$(\mathcal{L}_Y \mathcal{L}_X f)(p) - (\mathcal{L}_X \mathcal{L}_Y f)(p) = df(p)[dX(p)Y(p) - dY(p)X(p)] = (\mathcal{L}_{[X,Y]} f)(p)$$

3. The aim of this exercise is to prove that for every derivation $\delta : \mathcal{F}(M) \to \mathcal{F}(M)$, there exists a unique smooth vector field X such that $\delta = \mathcal{L}_X$.

a) Let $f : B_\epsilon(a) \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Prove the formula

$$f(x) = f(a) + \sum_{i=1}^n (x_i - a_i) g_i(x) \quad \text{with} \quad g_i(x) = \int_0^1 (\partial_{x_i} f)(a + t(x-a))\, dt.$$

b) Let δ be a derivation and $f \in \mathcal{F}(M)$. Prove the following

(i) **(Constants)** If $f \equiv c$ is constant, then $\delta(f) = 0$.

(ii) **(Localisation)** If $p \notin \mathrm{supp}(f)$, then $\delta(f)(p) = 0$.

(iii) **(Vanishing)** If $df(p) = 0$, then $\delta(f)(p) = 0$.

c) Use the properties of part b) to prove the statement.

Hint: (Localization) implies $\delta(f)(p) = \delta(g)(p)$ if f and g agree in a neighborhood of p. Hence we may assume for the proof of (**Vanishing**), that f is supported in a chart centred at p and then use the formula from a).

Solution:

a) Using the fundamental theorem of calculus and the chain rule, we obtain:

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \partial_t f(a + t(x-a))\, dt \\ &= \int_0^1 \sum_{i=1}^n (\partial_{x_i} f)(a + t(x-a))(x_i - a_i)\, dt \\ &= \sum_{i=1}^n (x_i - a_i) \int_0^1 (\partial_{x_i} f)(a + t(x-a))\, dt \end{aligned}$$

b) We prove **(i)**. The equation $\delta(1) = \delta(1 \cdot 1) = \delta(1) \cdot 1 + \delta(1) \cdot \delta(1) = 2\delta(1)$ implies $\delta(1) = 0$. For any constant $c \in \mathbb{R}$ follows by linearity $\delta(c) = c\delta(1) = 0$.

We prove **(ii)**. For $p \notin \mathrm{supp}(f)$ exists an open neighborhood $p \in U_0 \subset M$ such that $f(q) = 0$ for all $q \in U_0$. Let $\rho : M \to [0,1]$ be a smooth cutoff function with $\mathrm{supp}(\rho) \subset U_0$ and $\rho(p) = 1$. With $\eta := 1 - \rho$, it holds $f = f\eta$ and thus

$$\delta(f)(p) = \delta(f\eta)(p) = \delta(f)(p)\eta(p) + f(p)\delta(\eta)(p) = 0$$

as $f(p) = 0 = \eta(p)$.

We prove **(iii)**. Choose an open neighborhood $p \in U \subset M$ and a chart

$$\phi : B_\epsilon(0) \subset \mathbb{R}^m \to U \subset M$$

with $\phi(0) = p$. Let $\rho : U \to [0,1]$ be a smooth cutoff function with $\mathrm{supp}(\rho) \subset U$ and $\rho \equiv 1$ on a smaller open neighborhood $p \in U_0 \subset U$. Define $\tilde{f} = \rho f - f(p)$. It follows from **(i)** and **(ii)** that $\delta(f - \tilde{f})(p) = 0$ or equivalently

$$\delta(f)(p) = \delta(\tilde{f})(p).$$

By a) we can write $\tilde{f} \circ \phi$ as

$$\tilde{f} \circ \phi(x) = \sum_{i=1}^{n} h_i(x) g_i(x)$$

where $h_i(x), g_i(x)$ are smooth functions satisfying $h_i(0) = 0 = g_i(0)$. (Here we used the assumption $df(p) = 0$ or equivalently $d\tilde{f}(0) = 0$.) Hence

$$\begin{aligned}\delta(f)(p) = \delta(\tilde{f})(p) &= \sum_{i=1}^{n} \delta((h_i \circ \phi^{-1})(g_i \circ \phi^{-1}))(p) \\ &= \sum_{i=1}^{n} \delta((h_i \circ \phi^{-1}))(p) \cdot g_i(0) + h_i(0) \cdot \delta(g_i \circ \phi^{-1})(p) = 0.\end{aligned}$$

c) We argue first abstractly: Let $\Lambda : T_pM \to \mathbb{R}$ be a linear map and choose a smooth function $f \in \mathcal{F}(M)$ with $df(p) = \Lambda$. By **(Constants)** and **(Vanishing)** the value $\mathcal{X}_p(\Lambda) := \delta(f)(p)$ does not depend on the choice of f with $df(p) = \Lambda$. In particular, $\mathcal{X}_p$ defines an element in $(T_pM)^{**}$ which is canonically isomorphic to T_pM. Hence $\mathcal{X}$ describes a vector field $X \in \mathrm{Vect}(M)$ which satisfies $\mathcal{L}_X f = \delta(f)$.

Alternatively, we can work by **(Localization)** in local coordinates and assume $M = \Omega \subset \mathbb{R}^n$ is an open subset. Let $f_i(x) = x_i$ denote the coordinate functions and define

$$X : B_\epsilon(x) \to \mathbb{R}^n, \qquad X(x) = (\delta(f_1)(x), \ldots, \delta(f_n)(x)).$$

This is a smooth vector field on M and we claim that $\delta(f) = \mathcal{L}_X f$. Fix $a \in \Omega$, let $f : \Omega \to \mathbb{R}$ be a smooth function and let $\epsilon > 0$ be such that $B_\epsilon(a) \subset \Omega$. By part a), the following holds for $||x|| < \epsilon$:

$$f(x) = f(a) + \sum_{i=1}^{n} h_i(x) g_i(x)$$

with $h_i(x) = (x_i - a_i)$ and $g_i(x) = \int_0^1 \partial_{x_i} f(a + t(x-a))\, dt$. In particular, $h_i(a) = 0$, $g_i(a) = \partial_{x_i} f(a)$ and by **(Constants)** $\delta(h_i)(a) = X_i(a)$. Thus

$$\delta(f)(a) = \sum_{i=1}^{n} \delta(h_i)(a) \cdot g_i(a) + h_i(a) \cdot \delta(g_i)(a) = \sum_{i=1}^{n} X_i(a) \partial_{x_i} f(a) = df(a)X(a)$$

and this proves the claim.

4. Let M be a compact manifold. An automorphism of $\mathcal{F}(M)$ is a bijective $\mathbb{R}$-linear map $\Phi : \mathcal{F}(M) \to \mathcal{F}(M)$ satisfying

$$\Phi(fg) = \Phi(f)\Phi(g), \qquad \Phi(1) = 1.$$

The goal of this exercise is to show, that for every automorphism Φ there exists a diffeomorphism $\varphi : M \to M$ such that

$$\Phi(f) = f \circ \varphi =: \varphi^* f.$$

a) Show that every maximal ideal J of the ring $\mathcal{F}(M)$ has the shape

$$J_p = \{f \in \mathcal{F}(M) \,|\, f(p) = 0\}$$

for some $p \in M$.

b) Show that there exists a bijective map $\varphi : M \to M$ such that

$$\Phi^{-1}(J_p) = J_{\varphi(p)}, \qquad \Phi(J_p) = J_{\varphi^{-1}(p)},$$

and deduce from this that $\Phi(f) = f \circ \varphi$.

c) Suppose $\varphi : M \to M$ is a function such that $f \circ \varphi$ is smooth function for every $f \in \mathcal{F}(M)$. Show that φ is smooth.

Hint: For a): It is a result from abstract algebra that every ideal of a ring is contained in a maximal ideal. Now argue by contradiction: Suppose $J \subset \mathcal{F}(M)$ is an ideal and for every point $p \in M$ there exists a function $f_p \in J$ with $f(p) \neq 0$. By compactness there exists a finite linear combination $f = \sum_{i=1}^N f_{p_i}\rho_i$ which for suitable cutoff functions ρ_i has no zeros. Deduce from this the contradiction $J = \mathcal{F}(M)$.

Solution:

a) For $p \in M$ the map $\mathcal{F}(M)/J_p \to \mathbb{R}$, $[f] \mapsto f(p)$, is a well-defined isomorphism. In particular, $\mathcal{F}(M)/J_p$ is a field and therefore J_p a maximal ideal of $\mathcal{F}(M)$.

Conversely, let $J \subset \mathcal{F}(M)$ be a maximal ideal and assume that there exists $p \in M$ such that $f(p) = 0$ for all $f \in J$. Then it follows that $J \subset J_p$ and by maximality $J = J_p$.

Finally assume that $J \subset \mathcal{F}(M)$ is an ideal and assume that for every $p \in M$ there exists $f_p \in J$ with $f_p(p) \neq 0$. We claim that $J = \mathcal{F}(M)$ and in particular J is not a maximal ideal. Since $f_p \in J$ if and only if $-f_p \in J$, we may assume $f_p(p) > 0$. Now choose open neighborhoods $p \in U_p \subset M$ such that $f_p(x) > 0$ for all $x \in U_p$. By compactness of M, there exists a finite collection of points $\{p_i\}_{i=1,2,\dots,N}$ in M such that

$$M = U_{p_1} \cup U_{p_2} \cup \cdots \cup U_{p_N}.$$

Let $\rho_i : M \to [0, 1]$ for $i \in \{1, 2, \dots, N\}$ be a partition of unity subordinate to this cover, i.e. $\mathrm{supp}(\rho_i) \subset U_i$ and $\sum_{i=1}^N \rho_i(x) = 1$ for every $x \in M$. Since J is an ideal we have

$$f(x) = \sum_{i=1}^N \rho_i(x) f_{p_i}(x) \in J$$

and by construction $f(x) > 0$ for all $x \in M$. Hence $1/f \in \mathcal{F}(M)$ is a well-defined smooth function and therefore $1 = (1/f) \cdot f \in J$. This implies $J = \mathcal{F}(M)$ and completes the proof.

b) Since Φ is an automorphism, it sends maximal ideals to maximal ideals. Hence there exists a unique function $\varphi : M \to M$ such that

$$\Phi^{-1}(J_p) = J_{\varphi(p)}.$$

Note that φ is automatically bijective with inverse function defined by the relation $J_{\varphi^{-1}(p)} = \Phi(J_p)$. For every $p \in M$, the automorphism Φ induces a quotient map

$$[\Phi]_p : \mathcal{F}(M)/J_{\varphi(p)} \to \mathcal{F}(M)/J_p.$$

Denote for $f \in \mathcal{F}(M)$ by $[f]_{\varphi(p)}$ and $[f]_p$ its equivalence class in $\mathcal{F}(M)/J_{\varphi(p)}$ and $\mathcal{F}(M)/J_p$ respectively. For the constant function 1 we have $\Phi(1) = 1$ and thus $[\Phi]_p([1]_{\varphi(p)}) = [1]_p$. By linearity, this implies

$$[\Phi]_p([f]_{\varphi(p)}) = [\Phi]_p(f(\varphi(p)) \cdot [1]_{\varphi(p)}) = f(\varphi(p)) \cdot [1]_p.$$

On the other hand, from the definition of the quotient map follows directly

$$[\Phi]_p([f]_{\varphi(p)}) = [\Phi(f)]_p = \Phi(f)(p) \cdot [1]_p.$$

This shows $\Phi(f)(p) = f(\varphi(p))$ for ever $p \in M$.

c) Let $p \in M \subset \mathbb{R}^\ell$ and consider the functions

$$f_i : M \to \mathbb{R}, \qquad f(x_1, \ldots, x_\ell) = x_i.$$

By assumption $\varphi_i(x) := f_i(\varphi(x))$ are smooth functions and therefore

$$\varphi : M \to M, \qquad \varphi(x) = (\varphi_1(x), \ldots, \varphi_\ell(x))$$

is smooth.

5. Let $V \subset \mathbb{R}^\ell$ be a linear subspace.

a) Show that there exists a unique matrix $\Pi \in \mathbb{R}^{\ell \times \ell}$ satisfying

$$\Pi = \Pi^2 = \Pi^\top, \qquad \mathrm{Im}(\Pi) = V.$$

b) Let Π be as in a). Show that $\mathbb{R}^\ell = \ker(\Pi) \oplus \mathrm{Im}(\Pi)$ and $\ker(\Pi) \perp \mathrm{Im}(\Pi)$ are orthogonal subspaces.

c) Suppose $D \in \mathbb{R}^{\ell \times n}$ is injective and $V = \mathrm{Im}(D)$. Then $D^\top D$ is invertible and the matrix Π from a) is given by

$$\Pi = D(D^\top D)^{-1} D^\top.$$

Solution:

a) Assume we give ourself Π with these properties. Then let $v \in V = \mathrm{Im}(\Pi)$. So there exists $w \in \mathbb{R}^\ell$ such that $v = \Pi w$ and hence

$$\Pi v = \Pi(\Pi w) = \Pi^2 w = \Pi w = v$$

shows that Π restricts to the identity along V, i.e. $\Pi|_V = \mathbb{1}_V$.

Let $v \in V^\perp$ be in the orthogonal complement of $V = \mathrm{Im}(\Pi)$. This is equivalent to

$$\langle v, \Pi w \rangle = 0 \quad \forall w \in \mathbb{R}^\ell, \qquad \Leftrightarrow \qquad \langle \Pi^\top v, w \rangle = 0 \quad \forall w \in \mathbb{R}^\ell$$

and hence $\Pi v = \Pi^\top v = 0$. This shows $V^\perp \subset \ker(\Pi)$.

We have thus shown that $\Pi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is the map $\Pi = \mathbb{1}_V \oplus 0_{V^\perp}$ and therefore uniquely determined. This also shows existence.

b) We have seen above that $\mathrm{Im}(\Pi) = V$ and $\ker(\Pi) = V^\perp$ are orthogonal subspaces. They are of complementary dimensions.

c) Suppose v is in the kernel of $D^\top D$. Then follows in particular

$$0 = v^\top D^\top D v = \langle Dv, Dv \rangle = ||Dv||^2.$$

Since D is injective, this implies $v = 0$ and $D^\top D$ is injective. Since $D^\top D \in \mathbb{R}^{n\times n}$ is a quadradic matrix, injectivity already implies that $D^\top D$ is invertible. Hence

$$\Pi := D(D^\top D)^{-1} D^\top \in \mathbb{R}^{\ell\times\ell}$$

is well-defined. It clearly satisfies $\Pi = \Pi^2 = \Pi^\top$. Moreover, since $D^\top D$ is bijective, $D^\top$ must be surjective and therefore $\mathrm{Im}(\Pi) = \mathrm{Im}(D) = V$. Hence Π satisfies all the properties required in a).

6. Consider the following subset of $\mathbb{C}^2$:

$$E_{\text{Möb}} = \left\{ (e^{\mathbf{i}t}, \zeta) \in S^1 \times \mathbb{C} \,:\, \zeta \in \mathbb{R}e^{\mathbf{i}t/2} \right\}.$$

a) Show that $E_{\text{Möb}}$ is a rank 1 vector bundle over S^1.

b) Does there exists a section $\sigma : S^1 \to E_{\text{Möb}}$ with $\sigma(z) \neq 0$ fÃ$\frac{1}{4}$r all $z \in S^1$?

Hint: In a) Theorem 2.6.8 might be useful.

Solution:

a) For $e^{\mathbf{i}t} \in S^1$ the fiber

$$(E_{\text{Möb}})_{e^{\mathbf{i}t}} := \left\{ \zeta \in \mathbb{C} \,:\, (e^{\mathbf{i}t}, \zeta) \in E_{\text{Möb}} \right\} = \mathbb{R}e^{\mathbf{i}t/2}$$

is a real one dimensional subspace of $\mathbb{C} \cong \mathbb{R}^2$. We claim that the orthogonal projection onto this fiber is given by the formula

$$\Pi(e^{\mathbf{i}t})z = \mathrm{Re}(ze^{-\mathbf{i}t/2})e^{\mathbf{i}t/2}.$$

For this one only needs to verify

$$\begin{aligned} \Pi(e^{\mathbf{i}t})z = z \qquad & \forall z \in (E_{\text{Möb}})_{e^{\mathbf{i}t}} = \mathbb{R}e^{\mathbf{i}t/2} \\ \Pi(e^{\mathbf{i}t})z = 0 \qquad & \forall z \in (E_{\text{Möb}})^\perp_{e^{\mathbf{i}t}} = \mathbf{i}\mathbb{R}e^{\mathbf{i}t/2} \end{aligned}$$

which are both immediate from the formula. Since $\Pi : S^1 \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}) \cong \mathbb{R}^{2\times 2}$, $z \mapsto \Pi(z)$, is smooth, it follows from Theorem 2.6.8. that $E_{\text{Möb}}$ is a vector bundle over S^1.

b) Let $\sigma : S^1 \to E_{\text{Möb}}$ be a section. Define $\zeta_1 \in (E_{\text{Möb}})_1 = \mathbb{R}$ by $\sigma(1) = (1, \zeta_1)$ and define $\lambda : (0, 2\pi) \to \mathbb{R}$ by $\sigma(e^{\mathbf{i}t}) = \left(e^{\mathbf{i}t}, \lambda(t)e^{\mathbf{i}t/2}\right)$. Continuity of σ implies

$$\lim_{t\to 0} \lambda(t) = \lim_{t\to 0} \lambda(t)e^{\mathbf{i}t/2} = \zeta_1 = \lim_{t\to 2\pi} \lambda(t)e^{\mathbf{i}t/2} = -\lim_{t\to 2\pi} \lambda(t).$$

Therefore, either $\zeta_1 = 0$ or $\lambda(t)$ has opposite signs as t approaches 0 and 2π. In the later case $\lambda(t)$ vanishes at some intermediate point by the intermediate value theorem. This proves that any section of $E_{\text{Möb}}$ vanishes at some point.

7. For $i = 1, 2$ let $E_i \subset M \times \mathbb{R}^{\ell_i}$ be vector bundles with natural projections $\pi_i : E_i \to M$. A smooth map $\Phi : E_1 \to E_2$ is called a vector bundle isomorphism if $\pi_2 \circ \Phi = \pi_1$ and for every $p \in M$ the maps $\Phi_p : (E_1)_p \to (E_2)_p$ defined by

$$(p, \Phi_p v) = \Phi(p, v), \qquad \forall v \in (E_1)_p.$$

are linear and bijective. The vector bundles E_1 and E_2 are called isomorphic, if there exists a vector bundle isomorphism $\Phi : E_1 \to E_2$.

Consider the vector bundles $E_{\text{Möb}}$, TS^1, $S^1 \times \mathbb{R}$ over S^1. Which of these bundles are isomorphic?

Solution: For $z \in S^1$, we have $T_z S^1 = \mathbf{i}\mathbb{R}z = \{\mathbf{i}tz \,|\, t \in \mathbb{R}\}$. Hence the tangent bundle is given by

$$TS^1 = \{(z, \mathbf{i}tz) \in \mathbb{C}^2 \,|\, z \in S^1,\, t \in \mathbb{R}\}$$

This is isomorphic to the trivial bundle and the description ot TS^1 suggests the following vector bundle isomorphism:

$$\Phi : S^1 \times \mathbb{R} \to TS^1, \qquad (z, t) \mapsto (z, \mathbf{i}tz).$$

It is immediate to check that this map satisfies all the required properties.

We claim that $E_{\text{Möb}}$ is not isomorphic to $S^1 \times \mathbb{R}$. Suppose by contradiction there exists an isomorphism

$$\Phi : S^1 \times \mathbb{R} \to E_{\text{Möb}}$$

and consider the section $\sigma : S^1 \to S^1 \times \mathbb{R}$, $\sigma(z) = (z, 1)$. It follows from the properties of a vector bundle isomorphism that $\tilde{\sigma} := \Phi \circ \sigma : S^1 \to E_{\text{Möb}}$ is a section which is nowhere vanishing. We proved in Exercise 6 b) that such a section does not exist and therefore $E_{\text{Möb}}$ is not isomorphic to $S^1 \times \mathbb{R}$.

The notion of isomorphic bundles is clearly an equivalence relation and hence TS^1 is not isomorphic to $E_{\text{Möb}}$.

8. Let $E \subset M \times \mathbb{R}^\ell$ be vector bundle over M of rank r.

a) Let $f : N \to M$ be a smooth function. Show that

$$f^*E := \{(p, v) \in N \times \mathbb{R}^\ell \,:\, v \in E_{f(p)}\}$$

is a vector bundle over N of rank r.

b) Show that

$$E^\perp := \{(p, v) \in M \times \mathbb{R}^\ell \,:\, v \in E_p^\perp\}$$

is a smooth vector bundle over M. What is its rank?

c) For $i = 1, 2$ let $E_i \subset M \times R^{\ell_i}$ be vector bundles over M of rank r_i. Show that

$$E_1 \oplus E_2 := \{(p, v_1, v_2) \,:\, p \in M,\, v_1 \in (E_1)_p,\, v_2 \in (E_2)_p\} \subset M \times \mathbb{R}^{\ell_1 + \ell_2}$$

is a vector bundle over M of rank $r_1 + r_2$.

Hint: Use Theorem 2.6.8. to verify that these spaces are again vector bundles.

Solution:

a) Let $\Pi(q) \in \mathbb{R}^\ell$ be the orthogonal projection of $\mathbb{R}^\ell$ onto E_q. By Theorem 2.6.8 $\Pi : M \to \mathbb{R}^\ell$, $q \mapsto \Pi(q)$, is smooth. Hence the composition

$$\Pi \circ f : N \to \mathbb{R}^\ell$$

is also smooth and $(\Pi \circ f)(p)$ is the orthogonal projection onto $E_{f(p)} = (f^*E)_p$. Hence f^*E is a vector bundle by Theorem 2.6.8.

b) Let $\Pi : M \to \mathbb{R}^{\ell\times\ell}$ be as above. Then

$$\mathbb{1} - \Pi : M \to \mathbb{R}^{\ell\times\ell}, \qquad q \mapsto \mathbb{1} - \Pi(q)$$

is also smooth and $\mathbb{1} - \Pi(q)$ is the orthogonal projection onto $\ker(\Pi(q)) = (E_p)^\perp = (E^\perp)_p$. Hence $E^\perp$ is a vector bundle by Theorem 2.6.8. The rank of $E^\perp$ is the dimension of $(E_p)^\perp = (E^\perp)_p$ and thus equals $\ell - k$, where k is the rank of E.

c) Let $\Pi_i : M \to \mathbb{R}^{\ell_i\times\ell_i}$ be the smooth map of projections associated to E_1 and E_2. Then

$$\Pi = \Pi_1 \oplus \Pi_2 : M \to \mathbb{R}^{(\ell_1+\ell_2)\times(\ell_1+\ell_2)}, \qquad \Pi(q)(v_1, v_2) := (\Pi_1(q)v_1, \Pi_2(q)v_2)$$

is smooth and $\Pi(q)$ is the orthogonal projection onto $(E_1)_p \oplus (E_2)_p = (E_1 \oplus E_2)_p$. Hence $E_1 \oplus E_2$ is a vector bundle by Theorem 2.6.8.