Solution 5

1. Prove that every one-dimensional vector subbundle $E \subset TM$ is integrable.

Solution: We simply check that the subbundle is involutive and then invoke Frobenius' Theorem. Thus let $p \in M \subset \mathbb{R}^k$ and take a non-zero vector $v \in E_p$. By the theorem on vector bundles (at the time of writing 2.7.8), there is a vector field Z on M such that $Z(q) \in E_q$ for $q \in M$ and $Z(p) = v$. Then there is a neighbourhood $U \subset M$ of p such that $Z(q) \neq 0$ for all $q \in U$. So now let us do the test of involutivity at p. Given $X, Y : U \to \mathbb{R}^k$ with $X(q), Y(q) \in E_q$ for $q \in U$, then there are smooth functions f, g such that $X = fZ$ and $Y = gZ$ as L is one dimensional. So

$$[X,Y]_p = [fZ, gZ]_p = (df(p)Z(p) - dg(p)Z(p))Z(p) + f(p)g(p)[Z,Z]_p =: \lambda(p)Z(p) \in E_p,$$

where $\lambda(p) = df(p)Z(p) - dg(p)Z(p) \in \mathbb{R}$ is a scalar. As p was arbitrary, L is involutive.

Alternatively, one can argue directly without invoking Frobenius' Theorem: The flow lines of the vector field $Z(q)$ are one-dimensional submanifolds (at least locally) whose tangent spaces render the distribution E.

2. Consider the manifold $M = \mathbb{R}^3$. Prove that the subbundle $E \subset TM = \mathbb{R}^3 \times \mathbb{R}^3$ with fiber $E_p = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 : \zeta - y\xi = 0\}$ over $p = (x, y, z) \in \mathbb{R}^3$ is not integrable and that any two points in $\mathbb{R}^3$ can be connected by a path tangent to E.

Hint: Try to find two vector fields X, Y which span E at every point. Follow the flow lines of X and Y to find the path.

Solution: Define $X(p) = (0, 1, 0)$ and $Y(p) = (1, 0, y)$ for all $p = (x, y, z) \in \mathbb{R}^3$. It is easy to see that pointwise $X(p)$ and $Y(p)$ span E_p. However, $[X, Y]_p = (0, 0, -1) \notin E_p$ and so E is not involutive, and so by Frobenius' theorem cannot be integrable.

Now for the second statement, let us use the flows φ, ψ of X, Y respectively. These are given by

$$\varphi^t(x, y, z) = (x, y + t, z), \quad \psi^t(x, y, z) = (x + t, y, z + yt).$$

Now given a point $p = (x, y, z)$ which we want to connect to 0 by a path. First choose a smooth function $\rho : [0, 1] \to [0, 1]$ such that $\rho'(0) \geq 0$ and that there is $\epsilon > 0$ such that $\rho|_{[0,\epsilon)} \equiv 0$ and $\rho|_{(1-\epsilon,1]} \equiv 1$. Define $\gamma_p : [0, 4] \to \mathbb{R}^3$ by

$$\begin{cases} \gamma_p(t) = \psi^{(1-x)\rho(t)}(p), & \text{for } t \in [0,1], \\ \gamma_p(t) = \varphi^{(z-xy)\rho(t-1)}(\gamma_p(1)), & \text{for } t \in [1,2], \\ \gamma_p(t) = \psi^{-\rho(t-2)}(\gamma_p(2)), & \text{for } t \in [2,3], \\ \gamma_p(t) = \varphi^{(-y-z+xy)\rho(t-3)}(\gamma_p(3)), & \text{for } t \in [3,4]. \end{cases}$$

Then $\gamma_p(0) = p$, $\gamma_p(1) = (1, y, z + y(1-x))$, $\gamma_p(2) = (1, z + y(1-x), z + y(1-x))$, $\gamma_p(3) = (0, z + y(1-x), 0)$, $\gamma_p(4) = (0, 0, 0)$. The ρ in the formula is to ensure that we do a full stop at any turn, so that the resulting curve is smooth. To connect p to q, we can simply run from p to 0 via γ_p and then on to q by running through γ_q backward.

3. Let $M \subset \mathbb{R}^{m+1}$ be a submanifold of codimension one with a normal vector field $\nu : M \to \mathbb{R}^{n+1}$ i.e. $\nu(p) \perp T_pM$ for every $p \in M$ and $|\nu(p)| = 1$.

a) Give a formula for the second fundamental form in terms of ν.

b) Use this formula to derive the second fundamental form of $S^n \subset \mathbb{R}^{n+1}$.

Solution:

a) We use Exercises 5 c) and 8 b) of last week to conclude that in this case the tangent space orthogonal projection is given by

$$\Pi(p) = \mathbb{1} - D(p)(D(p)^\top D(p))^{-1} D(p)^\top = \mathbb{1} - \nu(p)\nu(p)^\top$$

for $p \in M$ where $D = \nu$ and so $D(p)^\top D(p) = |\nu(p)|^2 = 1$. And so for $v, w \in T_pM$, we get

$$h_p(v, w) = (d\Pi(p)v)w = -\nu(p)\langle d\nu(p)v, w\rangle - d\nu(p)v\langle \nu(p), w\rangle = -\nu(p)\langle d\nu(p)v, w\rangle.$$

as $\nu(p) \perp w$.

b) For the sphere, $\nu(p) = p$. So we get for $v, w \in T_pM$

$$h_p(v, w) = -p\langle v, w\rangle.$$

4. Let $p \in M \subset \mathbb{R}^k$ and $v \in T_pM$.

a) Prove that $(d\Pi(p)v)\xi \in T_pM$ for all $v \in T_pM$ and $\xi \in T_pM^\perp$.

b) Prove that the adjoint of $h_p(v, \cdot)$ is given by

$$h_p(v, \cdot)^* : T_pM^\perp \to T_pM, \qquad \xi \mapsto (d\Pi(p)v)\xi.$$

Hint: Consider the following setup for a): Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $\eta : (-\epsilon, \epsilon) \to TM^\perp$ be a smooth section with $\eta(t) \in T_{\gamma(t)}M$ and let $\xi \in T_pM^\perp$. Proceed by differentiating the equation $0 = \langle \Pi(\gamma(t))\xi, \eta(t)\rangle$.

Solution:

a) Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $\eta : (-\epsilon, \epsilon) \to TM^\perp$ be a section satisfying $\eta(t) \in T_{\gamma(t)}M^\perp$. For $\xi \in T_pM^\perp$ it follows $0 = \langle \Pi(\gamma(t))\xi, \eta(t)\rangle$ and hence

$$0 = \left.\frac{d}{dt}\right|_{t=0} \langle \Pi(\gamma(t))\xi, \eta(t)\rangle = \langle \Pi(p)\xi, \dot\eta(0)\rangle + \langle (d\Pi(p)v)\xi, \eta(0)\rangle = \langle (d\Pi(p)v)\xi, \eta(0)\rangle.$$

Because the normal bundle $TM^\perp$ is a vector bundle, we find for any initial vector $\eta_0 \in T_pM^\perp$ a smooth section $\eta : (-\epsilon, \epsilon) \to TM^\perp$ satisfying $\eta(t) \in T_{\gamma(t)}M^\perp$ and $\eta(0) = \eta_0$. Hence the calculation above implies that $(d\Pi(p)v)\xi$ is orthogonal to $T_pM^\perp$ and thus contained in T_pM.

b) The adjoint $h_p(v, \cdot)^* : T_pM^\perp \to T_pM$ is defined by the equation

$$\langle h_p(v, \cdot)^*\xi, w\rangle = \langle \xi, h_p(v, w)\rangle \qquad \forall w \in T_pM,\ \xi \in T_pM^\perp.$$

By part (a), we know that $(d\Pi(p)v)|_{T_pM^\perp} : T_pM^\perp \to T_pM$ is well-defined. In order to prove that it agrees with $h_p(v, \cdot)^* = (d\Pi(p)v)|_{T_pM^\perp}$ we need to show

$$\langle d\Pi(p)v)\xi, w\rangle = \langle \xi, h_p(v, w)\rangle, \qquad \forall w \in T_pM\, \forall \xi \in T_pM$$

Using the defnition of $h_p(v, w)$ this is equivalent to

$$\langle (d\Pi(p)v)\xi, w\rangle = \langle \xi, (d\Pi(p)v)w\rangle\,.$$

For this choose again a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Since $\Pi^\top(\gamma(t)) = \Pi(\gamma(t))$, we have $\langle \Pi(\gamma(t))\xi, w\rangle = \langle \xi, \Pi(\gamma(t))w\rangle$. Differentiating this identity yields

$$\langle (d\Pi(p)v)\xi, w\rangle = \frac{d}{dt}\bigg|_{t=0} \langle \Pi(\gamma(t))\xi, w\rangle = \frac{d}{dt}\bigg|_{t=0} \langle \xi, \Pi(\gamma(t))w\rangle = \langle \xi, (d\Pi(p)v)w\rangle$$

5. Let $M \subset \mathbb{R}^n$ be an m-manifold. Fix a point $p \in M$ and a unit tangent vector $v \in T_pM$ so that $|v| = 1$ and define

$$L := \{p + tv + w \,|\, t \in \mathbb{R}, w \perp T_pM\}.$$

Let $\gamma : (-\epsilon, \epsilon) \to M \cap L$ be a smooth curve such that $\gamma(0) = p, \dot\gamma(0) = v$, and $|\dot\gamma(t)| = 1$ for all t. Prove that

$$\ddot\gamma(0) = h_p(v, v).$$

Draw a picture of M and L in the case $n = 3$ and $m = 2$.

Hint: Write $\gamma(t) = p + \alpha(t)v + w(t)$ and show that $\ddot\gamma(0) = \ddot w(0) = h_p(v, v)$.

Solution: By definition of L, we can write the curve γ in the following way

$$\gamma(t) = p + \alpha(t)v + w(t)$$

where $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}$ and $w : (-\epsilon, \epsilon) \to T_pM^\perp$. Then

$$\dot\gamma(t) = \dot\alpha(t)v + \dot w(t) \quad \text{and} \quad \ddot\gamma(0) = \ddot\alpha(0)v + \ddot w(0).$$

Differentiating the assumption $1 = |\dot\gamma(t)|^2 = \dot\alpha(t)^2 + |\dot w(t)|^2$ yields

$$0 = \dot\alpha(t)\ddot\alpha(t) + \langle \dot w(t), \ddot w(t)\rangle.$$

Moreover, $\dot\gamma(0) = v$ yields $\dot\alpha(0) = 1$ and $\dot w(0) = 0$. Hence, at $t = 0$ the equation above yields $\ddot\alpha(0) = 0$ and therefore $\ddot\gamma(0) = \ddot w(0)$.

We show that $h_p(v, v) = \ddot w(0)$: Differentiating the identity $\Pi(\gamma(t))\dot\gamma(t) = \dot\gamma(t)$ yields

$$d\Pi(\gamma(t))\dot\gamma(t))\dot\gamma(t) + \Pi(\gamma(t))\ddot\gamma(t) = \ddot\gamma(t).$$

and we obtain at $t = 0$:

$$h_p(v, v) = d(\Pi(p)v)v = -\Pi(p)\ddot\gamma(0) + \ddot\gamma(0) = \ddot w(0).$$

The first equation is just the definition of the second fundamental form and the last equation follows from $\Pi(p)\ddot\gamma(0) = \ddot\alpha(0)$.

6. Choose a splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and write the elements of $\mathbb{R}^n$ as tuples $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$. Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold such that $p = 0 \in M$ and

$$T_0M = \mathbb{R}^m \times \{0\}, \quad T_0M^\perp = \{0\} \times \mathbb{R}^{n-m}.$$

By the implicit function theorem, there are open neighbourhoods $\Omega \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-m}$ of zero and a smooth map $f : \Omega \to V$ such that

$$M \cap (\Omega \times V) = \mathrm{graph}(f) = \{(x, f(x)) : x \in \Omega\}.$$

Thus $f(0) = 0$ and $df(0) = 0$. Prove that the second fundamental form $h_p : T_pM \times T_pM \to T_pM^\perp$ is given by the second derivatives of f, i.e.

$$h_p(v, w) = \left(0, \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right)$$

for all $v, w \in T_pM = \mathbb{R}^m \times \{0\}$.

Hint: Consider the vector fields $X(x,y) = (x, df(x)v)$ and $Y(x,y) = (x, df(x)w)$ and use that $h_p(v,w)$ agrees with the vertical component of $dX(p)Y(p)$.

Solution: Locally around the origin, the manifold M is the graph of f and its tangent spaces are

$$T_{(x,f(x))}M = \{(\hat{x}, df(x)\hat{x}) \,|\, \hat{x} \in \mathbb{R}^m\} \subset \mathbb{R}^m \oplus \mathbb{R}^{n-m}.$$

Let $v, w \in \mathbb{R}^m$ and define the vector fields $X, Y \in \mathrm{Vect}(M)$ by $X(x,y) = (v, df(x)v)$ and $Y(x,y) = (w, df(x)w)$. At $p = (0,0)$, we calculate

$$dX(p)Y(p) = \left(0, \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right) \in T_pM^\perp.$$

Since $X(0) = (v,0)$ and $Y(0) = (w,0)$, it follows that $h_p(v,w)$ is the orthogonal projection of $dX(p)Y(p)$ onto $T_pM^\perp$. For our choice of vector fields $dX(p)Y(p)$ is already contained in the orthogonal complement and thus

$$h_p(v,w) = dX(p)Y(p) = \left(0, \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right)$$