Solution 6

1. Recall that $\mathcal{O}(M)$ is the orthonormal frame bundle which as a set is equal

$$\mathcal{O}(M) := \{(p,e) \in \mathbb{R}^n \times \mathbb{R}^{n\times m} : p \in M, \operatorname{im} e = T_pM, e^\top e = \mathbb{1}_{m\times m}\}.$$

a) Prove that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$.

b) Let $(p,e) \in \mathcal{O}(M)$. What is the tangent space $T_{(p,e)}\mathcal{O}(M)$ and what is its dimension?

c) Prove that the map $\pi : \mathcal{O}(M) \to M$ is a submersion.

d) Prove that the action of $GL(m)$ on $\mathcal{F}(M)$ restricts to an action of the orthogonal group $O(m)$ on $\mathcal{O}(M)$ whose orbits are the fibers

$$\mathcal{O}(M)_p := \{e \in \mathbb{R}^{n\times m} : (p,e) \in \mathcal{O}(M)\} = \{e \in \mathcal{L}_{iso}(\mathbb{R}^m, T_pM) : e^T e = \mathbb{1}\}$$

Hint: In a), try to see $\mathcal{O}(M) \subset \mathcal{F}(M)$ as the level set of a regular value.

Solution:

a) Consider the map $f : \mathcal{F}(M) \to \mathrm{Sym}(m) = \{A \in \mathbb{R}^{m\times m} : A = A^T\}$ defined by

$$f(p,e) = e^T e.$$

In order to show that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$ it is enough to show that $\mathbb{1}$ is a regular value of f. Thus we have to check that $df(p,e)$ is surjective whenever $e^T e = \mathbb{1}$. To find $df(p,e)$ take a curve $(p(t), e(t)) \in \mathcal{F}(M)$ such that $p(0) = p$, $e(0) = e$ and $(\dot p(0), \dot e(0)) = (\hat p, \hat e)$. Then

$$df(p,e)(\hat p, \hat e) = \frac{d}{dt}\bigg|_{t=0} e^T(t)e(t) = \hat e^T e + e^T \hat e.$$

Recall from the lecture that

$$V_{(p,e)} = \{(0,\hat e) \,:\, \hat e \in \mathcal{L}(\mathbb{R}^m, T_pM)\} \subset T_{(p,e)}\mathcal{F}(M).$$

Now let $Y \in \mathrm{Sym}(m)$ be given. The equation $\hat e^T e + e^T \hat e = Y$ has a solution in $V_{(p,e)}$, namely $\hat e = \frac{1}{2} eY$. In particular, $\mathbb{1}$ is a regular value of f and it follows that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$.

b) It follows from the description of $\mathcal{O}(M)$ as a regular level set that

$$T_{(p,e)}\mathcal{O}(M) = \mathrm{Ker}(df(p,e)) = \{(\hat p, \hat e) \in T_{(p,e)}\mathcal{F}(M) :\ \hat e^T e + e^T \hat e = 0\}.$$

It follows most easily from the explicit description of $T_{(p,e)}\mathcal{O}(M)$ in Exercise 2 that $T_{(p,e)}\mathcal{O}(M)$ is a $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ dimensional vector space.
Here we can use the general formula for the dimension of a level set: $\dim(\mathcal{O}(M)) = \dim(\mathcal{F}(M)) - \dim(\mathrm{Sym}(m)) = m^2 + m - \frac{m(m+1)}{2} = \frac{m(m+1)}{2}$.

c) To prove that

$$\pi : \mathcal{O}(M) \to M, \qquad \pi(p,e) = p$$

is a submersion we have to prove that $d\pi(p,e) : T_{(p,e)}\mathcal{O}(M) \to T_pM$ is surjective. For this let $v \in T_pM$ be given and let $\gamma(t)$ be a smooth curve with $\gamma(0) = p$, $\dot\gamma(0) = v$. Define $e(t) = \Phi_\gamma(t,0)e$ using parallel transport and consider $\beta(t) = (\gamma(t), e(t)) \in \mathcal{O}(M)$. (Note that $\beta(t) \in \mathcal{O}(M)$ as parallel transport maps defines an orthonormal map $\Phi_\gamma(t,0) : T_pM \to T_{\gamma(t)}M$). Then

$$d\pi(p,e)(v,\dot e(0)) = \frac{d}{dt}\Big|_{t=0} \pi(\beta(t)) = \frac{d}{dt}\Big|_{t=0} \gamma(t) = v.$$

and this proves surjectivity.

d) The fiber $\mathcal{O}(M)_p$ is by definition

$$\mathcal{O}(M)_p := \{e \in \mathcal{L}_{iso}(\mathbb{R}^m, T_pM) \,:\, e^Te = \mathbb{1}\}.$$

The group $O(m)$ acts contavariantly on $\mathcal{O}(M)$ by

$$\mathcal{O}(M) \times O(m) \to \mathcal{O}(M) : ((p,e),g) \mapsto (p, g^*e) = (p, eg).$$

This is well defined, since $(eg)^T \cdot eg = g^Te^Teg = g^Tg = \mathbb{1}$ for $g \in O(m)$. The action is free, since $e = g * e = eg$ implies $\mathbb{1} = e^Te = e^Teg = g$. To see that it is transitive on each fiber let $e_0, e_1 \in \mathcal{O}(M)_p$ be given and define $g := e_0^Te_1 \in \mathbb{R}^{m\times m}$. Then $g^Tg = e_1^Te_0e_0^Te_1 = e_1^Te_1 = \mathbb{1}$ shows $g \in O(m)$ and one checks directly $e_1 = g^*e_0$.

2. Prove that $H_{(p,e)} \subset T_{(p,e)}\mathcal{O}(M)$ and that

$$T_{(p,e)}\mathcal{O}(M) = H_{(p,e)} \oplus V'_{(p,e)}; \;\; V'_{(p,e)} = V_{(p,e)} \cap T_{(p,e)}\mathcal{O}(M),$$

for every $(p,e) \in \mathcal{O}(M)$.

Solution: We have seen in Exercise 1 that

$$T_{(p,e)}\mathcal{O}(M) = \{(\hat p, \hat e) \in T_{(p,e)}\mathcal{F}(M) \,:\, e^T\hat e + \hat e^Te = 0\}.$$

And we have seen in the lecture that $T_{(p,e)}\mathcal{F}(M) = H_{(p,e)} \oplus V_{(p,e)}$ decomposes as the direct sum of a horizontal and vertical space which are given by

$$H_{(p,e)} = \{(v, h_p(v)e) \,:\, v \in T_pM\}, \qquad V_{(p,e)} = \{(0,\hat e) \,:\, \hat e \in \mathcal{L}(\mathbb{R}^m, T_pM)\}.$$

We prove first $H_{(p,e)} \subset T_{(p,e)}\mathcal{O}(M)$: It holds

$$\langle e\xi, h_p(v)e\hat p\rangle = 0 \qquad \forall \xi \in \mathbb{R}^m,\, \forall \hat p \in T_pM$$

since $e\xi \in T_pM$ and $h_p(v)e\hat p \in T_pM$. Therefore $e^Th_p(v) = 0$ and hence $e^Th_p(v)ev + (h_p(v)ev)^Te = 0$ shows $(v, h_p(v)e) \in T_{(p,e)}\mathcal{O}(M)$.

The vertical space of $\mathcal{O}(M)$ is given as

$$V'_{(p,e)} = T_{(p,e)}\mathcal{O}(M) \cap V_{(p,e)} = \{(0,\hat e) \,:\, \hat e \in \mathcal{L}(\mathbb{R}^m, T_p) \text{ satisfying } \hat e^Te + e^T\hat e = 0\}$$

The last equation says that $e^T\hat e \in \mathrm{Asym}(m)$ is a anti-symmetric matrix. Conversely, every anti-symmetric matrix $a \in \mathrm{Asym}(m)$ determines a vertical tangent vector with $\hat e = ea$.

We clearly have $T_{(p,e)}\mathcal{O}(M) = H_{(p,e)} \oplus V'_{(p,e)}$. Moreover, $\dim(H_{(p,e)}) = m$ and $\dim(V'_{(p,e)}) = \dim(\mathrm{Asym}(m)) = \frac{m(m-1)}{2}$ shows $\dim(T_{(p,e)}\mathcal{O}(M)) = m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$.

3. **[Parallel transport on S^2]** Let $\alpha \in [0, 2\pi]$. Consider the following curves on S^2

$$\begin{aligned}
\gamma_1(t) &:= (\sin(t), 0, \cos(t)), \quad t \in [0, \pi/2], \\
\gamma_2(t) &:= (\cos(t), \sin(t), 0), \quad t \in [0, \alpha], \\
\gamma_3(t) &:= (\cos(t)\cos(\alpha), \cos(t)\sin(\alpha), \sin(t)), \quad t \in [0, \pi/2].
\end{aligned}$$

a) Draw the images of γ_i, $i = 1, 2, 3$.

b) Let $v \in T_{(0,0,1)}S^2$. What is the parallel transport of v along γ_1, i.e. $\Phi_{\gamma_1}(t, 0)v$?

c) Let γ be the piecewise smooth curve, which we obtain by going along γ_1, then γ_2 and at the end γ_3. What is the parallel transport along γ, i.e. $\Phi_\gamma(\pi + \alpha, 0)v$, for $v \in T_{(0,0,1)}S^2$.

Hint: Think of parallel transport as a mighty soldier carrying his lance in front of him as he walks along the curves without ever turning at the corners.

Solution:

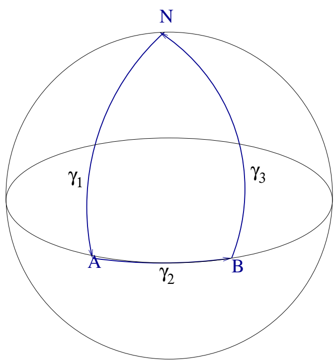

Figure 1: Curves γ_i, $i = 1, 2, 3$

a)

b) Let $v \in T_{(0,0,1)}S^2 = T_N S^2$. The tangent space at the north pole $T_N S^2$ is generated by the vectors $v_1 = \dot\gamma_1(0) = (1, 0, 0)$ and $v_2 = (0, 1, 0)$. It is enough to find $\Phi_{\gamma_1}(t, 0)v_i$ for $i = 1, 2$ as parallel transport in linear: I.e. for an arbitrary $v = \alpha v_1 + \beta v_2 \in T_N S^2$ it holds

$$\Phi_{\gamma_1}(t, 0)v = \alpha\Phi_{\gamma_1}(t, 0)v_1 + \beta\Phi_{\gamma_1}(t, 0)v_2.$$

We observe first that $v_1(t) = \dot\gamma_1(t)$ is a parallel vector field along $\gamma_1(t)$: Since $\dot v_1(t) = \ddot\gamma(t) = (-\sin(t), 0, -\cos(t)) \in (T_{\gamma(t)}S^2)^\perp$ its orthogonal projection onto $T_{\gamma(t)}S^2$ vanishes and thus $\dot v_1(t)$ is parallel. This yields

$$\Phi_{\gamma_1}(t, 0)v_1 = \dot\gamma_1(t) = (\cos(t), 0, -\sin(t)).$$

For v_2, the constant vector field

$$v_2(t) := \Phi_{\gamma_1}(t, 0)v_2 = (0, 1, 0)$$

does the job. (By Definition $v_2(t)$ is parallel if the projection of $\dot v_2(t)$ onto $T_{\gamma(t)}S^2$ vanishes. But we have already $\dot v(t) = 0$).

We thus obtain for $v = \alpha v_1 + \beta v_2 = (\alpha, \beta, 0)$:

$$\Phi_{\gamma_1}(t, 0)v = (\alpha\cos(t), \beta, -\alpha\sin(t)).$$

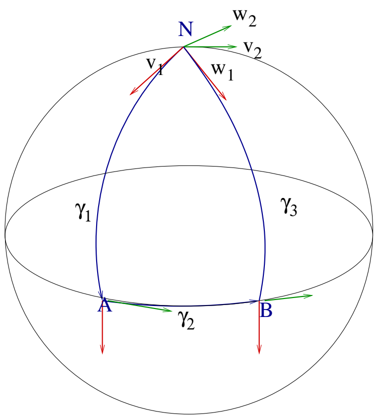

Figure 2: Parallel transport along γ

c) First notice that the sphere triangle $\triangle NAB$ has two sides of length $\frac{\pi}{2}$ and one of α and that parallel transport along γ is defined by

$$\Phi_\gamma(\pi+\alpha,0):T_NS^2\to T_NS^2,\quad \Phi_\gamma(\pi+\alpha,0)=\Phi_{\gamma_3}(\pi/2,0)\circ\Phi_{\gamma_2}(\alpha,0)\circ\Phi_{\gamma_1}(\pi/2,0)$$

The parallel transport along γ_2 and γ_3 can be calculated in the same way as in part b): In both cases $\dot\gamma_2$ and $\dot\gamma_3$ are parallel and rotating these vector fields by $\pi/2$ we obtain constant and hence parallel vector fields. In summary, we get for $v_1=(1,0,0)$ and $v_2=(0,1,0)$:

$$\Phi_\gamma(\pi+\alpha,0)v_1=\Phi_{\gamma_3}(\pi/2,0)\circ\Phi_{\gamma_2}(\alpha,0)\circ\Phi_{\gamma_1}(\pi/2,0)v_1=(\cos(\alpha),\sin(\alpha),0),$$
$$\Phi_\gamma(\pi+\alpha,0)v_2=\Phi_{\gamma_3}(\pi/2,0)\circ\Phi_{\gamma_2}(\alpha,0)\circ\Phi_{\gamma_1}(\pi/2,0)v_2=(-\sin(\alpha),\cos(\alpha),0).$$

By linearity, this defines $\Phi_\gamma(\pi+\alpha,0)$ completely.

4. The first fundamental from on a manifold $M^m\subset\mathbb{R}^n$ is defined by

$$g:TM\oplus TM\to\mathbb{R},\ (p,v\oplus w)\mapsto g_p(v,w):=\langle v,w\rangle.$$

a) Let $\psi:\Omega\subset\mathbb{R}^m\to U\subset M$ be a local parametrisation. The functions $g_{ij}:\Omega\to\mathbb{R},\ i,j=1,\dots,m$ are defined by

$$g_{ij}(x):=\sum_{\nu=1}^n\frac{\partial\psi^\nu}{\partial x^i}(x)\frac{\partial\psi^\nu}{\partial x^j}(x).\tag{1}$$

How are the matrix $(g_{ij})_{i,j=1,\dots,n}$ and the first fundamental form g related?

b) Let $\varphi:S^n\setminus\{N\}\to\mathbb{R}^n$ be the stereographic projection. Find a formula for φ and its inverse $\psi:\mathbb{R}^n\to S^n\setminus\{N\}$. Find the metric $g_{ij},\ i,j=1,\dots,n$ in these local coordinates.

Solution:

a) Take the vector fields $X_i=\frac{\partial\psi}{\partial x^i}$ for $i=1,\dots,m$. Then any vector field $X:U\to\mathbb{R}^n$ of M can be written as $X(\psi(x))=\sum_{i=1}^m\xi^i(x)X_i(x)$ for $x\in\Omega$ and some functions $\xi_i:\Omega\to\mathbb{R}$. Similarly, $Y(\psi(x))=\sum_{i=1}^m\eta^i(x)X_i(x)$. Thus we get

$$g_{\psi(x)}(X(\psi(x)),Y(\psi(x)))=\sum_{i,j=1}^m\xi^i\eta^jg_{\psi(x)}(X_i(x),X_j(x))=\sum_{i,j=1}^m\xi^i\eta^jg_{ij}(x).$$

So g_{ij} are the coefficients of the metric g in local coordinates.

b) If we denote $S^n = \left\{(y_1, \cdots, y_{n+1}) \in \mathbb{R}^{n+1} \middle| \sum_{i=1}^{n+1} y_i^2 = 1\right\}$ and let $(x_1, x_2, \cdots, x_n)$ be the coordinates in $\mathbb{R}^n$ we have

$$\varphi : S^n \setminus \{N\} \to \mathbb{R}^n, \quad \varphi(y) = x = (x_1, \cdots, x_n), \quad x_i = \frac{y_i}{1 - y_{n+1}}.$$

The inverse map $\psi = \varphi^{-1} : \mathbb{R}^n \to S^n \setminus \{N\}$ is given by

$$\psi(x_1, \cdots, x_n) = \frac{1}{1 + |x|^2}(2x_1, 2x_2, \cdots, |x|^2 - 1).$$

Thus we get the formulae for $\nu = 1, 2, \ldots, n$ that

$$\frac{\partial \Psi^\nu}{\partial y^i}(y) = \frac{2\delta_{i\nu}(|y|^2 + 1) - 4y_i y_\nu}{(|y|^2 + 1)^2}, \qquad \frac{\partial \Psi^{n+1}}{\partial y^i}(y) = \frac{4y_i}{|y|^2 + 1}$$

So for the metric we get for $i, j = 1, \ldots, n$

$$\begin{aligned} g_{ij} &= \sum_{\nu=1}^{n+1} \frac{\partial \psi^\nu}{\partial x^i}\frac{\partial \psi^\nu}{\partial x^j} = \sum_{\nu=1}^{n} \left(\frac{2\delta_{i\nu}(|y|^2 + 1) - 4y_i y_\nu}{(|y|^2 + 1)^2}\right)\left(\frac{2\delta_{j\nu}(|y|^2 + 1) - 4y_j y_\nu}{(|y|^2 + 1)^2}\right) + \frac{16 y_i y_j}{(|y|^2 + 1)^4} \\ &= \frac{1}{(1 + |y|)^4}\left(4\delta_{ij}(|y|^2 + 1)^2 - 8\sum_{i=1}^{n} y_\nu(y_j \delta_{i\nu} + y_i \delta_{j\nu})(|y|^2 + 1) + 16 y_i y_j (|y|^2 + 1)\right) \\ &= \frac{4}{(1 + |y|^2)^2}\delta_{ij}. \end{aligned}$$

5. Let $M^m \subset \mathbb{R}^n$ be a manifold. Let $\psi : \Omega \to U \subset M$ be a local parametrisation. Let $c : I \to \Omega$ be a curve and define $\gamma := \psi \circ c$. Any vector field $X : I \to \mathbb{R}^n$ along γ can be written as $X(t) = \sum_{k=1}^m \xi^k(t) \frac{\partial \psi}{\partial x^k}(c(t))$.

Define the **Christoffel symbols** $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ for $i, j, k = 1, \ldots, m$ by

$$\Pi(\psi(x)) \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x) =: \sum_{k=1}^{m} \Gamma_{ij}^k(x) \frac{\partial \psi}{\partial x^k}(x).$$

a) The vector field $\nabla X : I \to \mathbb{R}^n$ is defined by $\nabla X(t) = \Pi(\gamma(t))\dot{X}(t)$ for all $t \in I$. This can be expressed as $\nabla X(t) = \sum_{k=1}^m \eta^k(t) \frac{\partial \psi}{\partial x^k}(c(t))$. What is the relationship between η^k, ξ^k and Γ_{ij}^k?

b) Define the matrix $(g^{ij}(x))$ to be the inverse of $(g_{ij}(x))$ for $x \in \Omega$. Define $\Gamma_{kij} : \Omega \to \mathbb{R}$ for $i, j, k \in 1, \ldots, m$ by $\Gamma_{ij}^\ell = \sum_{k=1}^m g^{\ell k} \Gamma_{kij}$. Prove that for $i, j, k \in 1, \ldots, m$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ijk} + \Gamma_{jik}.$$

c) Prove that

$$\Gamma_{kij} = \Gamma_{kji}, \quad \text{and} \quad \Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

This last equation says that the Christoffel symbols are ‘intrinsic’.

d) Use a) to find another proof of short time existence for parallel transport along a curve γ as in Theorem 1 (3.1.16. in the lecture notes). I.e. prove that for $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$, there is $\epsilon > 0$ such that $I_0 := (t_0 - \epsilon, t_0 + \epsilon) \subset I$ and such that there is a unique $X \in \mathrm{Vect}(\gamma|_{I_0})$ such that $\nabla X = 0$ and $X(t_0) = v_0$.

Hint: For b), use $\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle$ for suitable $X, Y \in \mathrm{Vect}(\gamma)$ as a starting point. In c) use the symmetry (1. equation) and b) to get to the 2. equation. Think of how you would integrate $e^t \cos(t)$.

Solution:

a) We calculate

$$\begin{aligned}\nabla X(t) &= \pi(\gamma(t))\frac{d}{dt}\left(\sum_{k=1}^m \xi^k(t)\frac{\partial\psi}{\partial x^k}(c(t))\right)\\ &= \pi(\gamma(t))\sum_{k=1}^m \dot\xi^k\frac{\partial\psi}{\partial x^k}(c(t)) + \pi(\gamma(t))\sum_{i,j=1}^m \frac{\partial^2\psi}{\partial x^i\partial x^j}(c(t))\xi^i(t)\dot c^j(t)\\ &= \sum_{k=1}^m \underbrace{\left(\dot\xi^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c(t))\xi^i\dot c^j(t)\right)}_{=\eta^k}\frac{\partial\psi}{\partial x^k}(c(t)).\end{aligned}$$

b) We define $X_i(t) = \frac{\partial\psi}{\partial x^i}(c(t))$ with coefficients $\xi^j = \delta_{ij}$ for $i, j = 1, \dots, m$. From a), we get $\nabla X_i(t) = \sum_{j,k=1}^m \Gamma_{ij}^k \dot c^j X_k(c(t))$. Also $g_{ij}(c(t)) = \langle X_i(t), X_j(t)\rangle$. So let us choose $c(t) = x_0 + te_k$ where e_k is the k^{th} standard basis vector of $\mathbb{R}^m$ and therefore $\dot c^j(t) = \delta_{jk}$. And calculate

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k} &= \frac{d}{dt}\langle X_i(t), X_j(t)\rangle = \langle\nabla X_i(t), X_j(t)\rangle + \langle X_i(t), \nabla X_j(t)\rangle\\ &= \sum_{\ell=1}^m(\Gamma_{ik}^\ell g_{\ell j} + g_{i\ell}\Gamma_{jk}^\ell) = \Gamma_{jik} + \Gamma_{ijk}.\end{aligned}$$

c) The symmetry in the first equation is due to $\frac{\partial^2\psi}{\partial x^i\partial x^j} = \frac{\partial^2\psi}{\partial x^j\partial x^i}$ for all $i, j = 1, \dots, m$.

For the second equation, we will do 'integration by part'. We calculate

$$\begin{aligned}\Gamma_{kij} &= \frac{g_{ki}}{\partial x^j} - \Gamma_{ikj} = \frac{g_{ki}}{\partial x^j} - \Gamma_{ijk} = \frac{g_{ki}}{\partial x^j} - \frac{g_{ij}}{\partial x^k} + \Gamma_{jik}\\ &= \frac{g_{ki}}{\partial x^j} - \frac{g_{ij}}{\partial x^k} + \Gamma_{jki} = \frac{g_{ki}}{\partial x^j} - \frac{g_{ij}}{\partial x^k} + \frac{g_{jk}}{\partial x^i} - \Gamma_{kji}\\ &\Rightarrow \Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).\end{aligned}$$

d) For a curve $\gamma : I \to M$ and $t_0 \in I$, we may restrict the interval I to a smaller interval around t_0 such that $\operatorname{im}\gamma \subset U$. Take $c : I \to \Omega$ given by $c = \psi^{-1}\circ\gamma$. Then the equation $\nabla X = 0$ means that all coefficients η^k from a) have to be simultaneously zero. Thus the parallel $X = \sum_{i=1}^m \xi^i X_i(c(t))$ is given as the solution of the linear system of ODE's subject to an initial condition $v_0 =: \sum_{i=1}^m v_0^k X_k(\psi^{-1}(\gamma(t_0)))$

$$\begin{cases}\dot\xi^k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(c(t))\xi^i(t)\dot c^j(t) = 0, & \text{for } k = 1, \dots, m,\\ \xi^k(0) = v_0^k, & \text{for } k = 1, \dots, m.\end{cases}$$

From Analysis II, we know that such a system with initial condition has a unique solution $(\xi^1, \xi^2, \dots, \xi^m) : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^m$ for some $\epsilon > 0$.

6. Consider the case $m = 2$. Let $\Omega \subset \mathbb{R}^2$ be an open set and $\lambda : \Omega \to (0, +\infty)$ be a smooth function. Suppose that the metric $g : \Omega \to \mathbb{R}^{2\times 2}$ is given by

$$g(x) = \begin{pmatrix}\lambda(x) & 0\\ 0 & \lambda(x)\end{pmatrix}.$$

a) Find the Christoffel symbols Γ_{ij}^k by using the formula in Exercise 5 c).

b) For $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\} := \mathbb{H}$ and $\lambda(x_1, x_2) = \frac{1}{x_2^2}$, calculate Γ^k_{ij}.

$(\mathbb{H}, g)$ is called Poincaré half-plane model for two dimensional hyperbolic geometry.

Solution:

a) We have that $\Gamma^k_{ij} = g^{k\ell}\Gamma_{\ell ij}$ as the metric is diagonal, we have

$$\Gamma^k_{ij} = g^{kk}\Gamma_{kij} = \frac{1}{\lambda(x)}\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

We have to find Γ^k_{ij} for $i, j, k = 1, 2$. Using the fact that $g_{12} = g_{21} = 0$ and $g_{11} = g_{22} = \lambda(x)$ we get

$$\Gamma^1_{11} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial x^1}, \qquad \Gamma^2_{22} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial x^2},$$
$$\Gamma^2_{11} = \frac{-1}{2\lambda}\frac{\partial\lambda}{\partial x^2}, \qquad \Gamma^1_{22} = \frac{-1}{2\lambda}\frac{\partial\lambda}{\partial x^1},$$
$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial x^2}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial x^1}.$$

b) We get readily from a) that

$$\Gamma^2_{21} = \Gamma^2_{12} = \Gamma^1_{11} = 0, \quad \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{x_2}, \quad \Gamma^2_{11} = \frac{1}{x_2}.$$