Solution 7

1. **[Development of the cylinder Z]** Consider the cylinder $Z := \mathbb{R} \times S^1$ and let

$$\gamma : [0,1] \to Z, \qquad \gamma(t) = (h(t), e^{i\theta(t)})$$

be a smooth curve. Let $p_0 \in \mathbb{R}^2$ and $\Psi_0 : T_{\gamma(0)}Z \to \mathbb{R}^2$ be an orthogonal isomorphism.

a) Find the development (Ψ, γ, γ') of Z along $\mathbb{R}^2$ with $\gamma'(0) = p_0,\ \Psi(0) = \Psi_0$.

b) Give an example of a closed curve $\gamma : [0,1] \to Z,\ \gamma(0) = \gamma(1)$, such that γ' is not closed.

c) From this example give a geometric interpretation of the term development.

Hint: The path of orthogonal isometries $\Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t) : T_{\gamma(t)}Z \to \mathbb{R}^2$ does not depend on $\gamma'(t)$ (as parallel transport in $\mathbb{R}^2$ is trivial). Hence, one can first solve for $\Phi(t)$ and then recover $\gamma'(t)$ from the equation $\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$.

Solution:

a) Write $\gamma(t) = (h(t), e^{\mathbf{i}\theta(t)})$. The vectors $v_0 := (1,0)$ and $v_1 := (0, \mathbf{i}e^{\mathbf{i}\theta(0)})$ define an orthonormal basis of $T_{\gamma(0)}Z$. Parallel transport of these vectors along γ is given by

$$v_0(t) = \Phi_\gamma(t,0)v_0 = (1,0), \qquad v_1(t) = \Phi_\gamma(t,0)v_1 = (0, \mathbf{i}e^{\mathbf{i}\theta(t)}).$$

For this is suffices to note that $\dot v_0(t) = 0$ and $\dot v_1(t) = (0, -e^{\mathbf{i}\theta(t)}) \in T^\perp_{\gamma(t_0)}Z$ and so $v_0(t)$ and $v_1(t)$ are indeed parallel vector fields. This completely determines the map $\Phi_\gamma(t,0) : T_{\gamma(0)}Z \to T_{\gamma(t)}Z$ by linearity, i.e.

$$\Phi_\gamma(t,0)(\alpha, \mathbf{i}\beta e^{\mathbf{i}\theta(0)}) = (\alpha, \mathbf{i}\beta e^{\mathbf{i}\theta(t)})$$

and its inverse $\Phi_\gamma(0,t) : T_{\gamma(t)}Z \to T_{\gamma(0)}Z$ is given by

$$\Phi_\gamma(0,t)(\hat h, \mathbf{i}\hat\theta e^{\mathbf{i}\theta(t)}) = (\hat h, \mathbf{i}\hat\theta e^{\mathbf{i}\theta_0})$$

Since $\Phi'_{\gamma'}(t,0) : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity, it follows

$$\Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t) : T_{\gamma(t)}Z \to \mathbb{R}^2, \qquad \Phi(t)(\hat h, \mathbf{i}\hat\theta e^{\mathbf{i}\theta(t)}) = \hat h v_0' + \hat\theta v_1'$$

where $v_0' := \Phi_0 v_0$ and $v_1' := \Phi_0 v_1$. The equation $\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$ is thus equivalent to

$$\dot\gamma'(t) = \Phi(t)(\dot h(t), \mathbf{i}\dot\theta(t)e^{\mathbf{i}\theta(t)}) = \dot h(t)v_0' + \dot\theta(t)v_1'$$

and integration yields

$$\gamma'(t) = p_0 + \int_0^s \dot\gamma'(s)\,ds = p_0 + (h(t) - h(0))\,v_0' + (\theta(t) - \theta(0))v_1'.$$

b) Take the curve $\gamma : [0,1] \to Z$, $\gamma(t) := (0, e^{2\pi\mathbf{i}t})$. Then γ is closed and satisfies $\gamma(0) = (0,1) = \gamma(1)$. However, by the formula obtained in a) the curve $\gamma' : [0,1] \to \mathbb{R}^2$ is the straight line $\gamma(t) = p_0 + 2\pi t v_1'$ and certainly not a closed curve.

c) To develop a cylinder means to roll it along $\mathbb{R}^2$.

2. **[Development of S^2 along $\mathbb{R}^2$]** Let $\gamma : [0, \pi + \alpha] \to S^2$ be the curve from the exercise 3 c) of exercise sheet 6. Let $\Psi_0 : T_{(0,0,1)}S^2 \to T_0\mathbb{R}^2$ be the orthogonal isomorphism given by

$$\Psi_0\,(1,0,0) := (1,0) \quad \Psi_0\,(0,1,0) := (0,1)\,.$$

a) Find the development (Ψ, γ', γ) of S^2 along $\mathbb{R}^2$ with $\gamma'(0) = p_0'$, $\Psi(0) = \Psi_0$.

b) For which $\alpha \in [0, 2\pi)$ is the curve γ' in $a)$ closed?

Hint: Proceed as in Exercise 1 to calculate the developement. The curve γ' and Ψ will be just piecewise smooth and we require the condition $\Psi(t)\dot\gamma(t) = \dot\gamma'(t)$ to hold only for $t \neq \pi/2,\ \pi/2 + \alpha$.

Solution:

a) We have seen in exercise 3b) of exercise sheet 6 that the parallel transport along γ_1 is given by the formula

$$\Phi_{\gamma_1}(t, 0)(a, b, 0) = (a\cos(t), b, -a\sin(t))$$

Since parallel transport in $\mathbb{R}^2$ is the identity, it follows $\Phi_1(t) := \Phi_0\Phi_{\gamma_1}(0, t) : T_{\gamma_1(t)}S^2 \to \mathbb{R}^2$. This map is uniquely determined by the values

$$\Phi_1(t)\dot\gamma_1(t) = (1, 0), \qquad \Phi_1(t)(0, 1, 0) = (0, 1)$$

as $\dot\gamma_1(t)$ and $(0, 1, 0)$ form an orthonormal basis of $T_{\gamma_1(t)}S^2$. It follows that the curve $\gamma_1' : [0, \pi/2] \to \mathbb{R}^2$ satisfies

$$\dot\gamma_1'(t) = \Phi_1(t)\dot\gamma_1(t) = (1, 0)$$

and therefore $\gamma_1'(t) = (t, 0)$.

The initial conditions for the developement along γ_2 are $\gamma_2'(0) = \gamma_1'(\pi/2) = (\pi/2, 0)$ and $\Phi_2(0) = \Phi_1(\pi/2) : T_{(0,1,0)}S^2 \to \mathbb{R}^2$ is the orthogonal map

$$\Phi_2(0)(0, 1, 0) = (0, 1), \qquad \Phi_2(0)(0, 0, 1) = (-1, 0)$$

The parallel transport along γ_2 is given by the formula

$$\Phi_{\gamma_2}(t, 0)(0, a, b) = (-a\sin(t), a\cos(t), b)$$

We then obtain as above that $\Phi_2(t)$ is uniquely determined by

$$\Phi_2(t)\dot\gamma_2(t) = (0, 1), \qquad \Phi_2(t)(0, 0, 1) = (-1, 0).$$

Hence $\gamma_2' : [0, \alpha] \to \mathbb{R}^2$ satisfies

$$\dot\gamma_2'(t) = \Phi_2(t)\dot\gamma_2(t) = (0, 1)$$

and we obtain $\gamma_2'(t) = (\pi/2, t)$.

Finally, the initial conditions for γ_3' are $\gamma_3'(0) = \gamma_2'(\alpha) = (\frac{\pi}{2}, \alpha)$ and $\Phi_3(0) = \Phi_2(\alpha)T_{(\cos(\alpha),\sin(\alpha),0)}$ is uniquely determined by

$$\Phi_3(0)((-\sin(\alpha), \cos(\alpha), 0)) = (0, 1), \qquad \Phi_3(0)(0, 0, 1) = (-1, 0).$$

Parallel transport along γ_3 satisfies

$$\Phi_{\gamma_3}(t, 0)(\cos(\alpha), \sin(\alpha), 0) = (-\sin(\alpha), \cos(\alpha), 0)$$
$$\Phi_{\gamma_3}(t, 0)(0, 0, 1) = (-\sin(t)\cos(\alpha), -\sin(t)\cos(\alpha), \cos(t)).$$

Geometrically, this exactly the same situation as before and the map $\Phi_3(t)$ is uniquely determined by

$$\Phi_3(t)\dot{\gamma}_3(t) = (-1, 0), \qquad \Phi_3(t)(-\sin(\alpha), \cos(\alpha), 0) = (0, 1).$$

Hence the curve $\gamma_3' : [0, \pi/2] \to \mathbb{R}^2$ satisfies

$$\dot{\gamma}_3'(t) = \Phi_3(t)\dot{\gamma}_3(t) = (-1, 0)$$

and we obtain $\gamma_3'(t) = (\pi/2 - t, \alpha)$.

Concatenating these three developements yields the developement along γ. In particular

$$\gamma'(t) = \begin{cases} (t, 0) & \text{for } 0 \le t \le \pi/2 \\ (\pi/2, t - \pi/2) & \text{for } \pi/2 \le t \le \pi/2 + \alpha \\ (\pi + \alpha - t, \alpha) & \text{for } \pi/2\alpha \le t \le \pi + \alpha \end{cases}$$

b) The formula in a) yields $\gamma'(0) = (0, 0)$ and $\gamma'(\pi + \alpha) = (0, \alpha)$. Hence $\gamma' : [0, \pi + \alpha] \to \mathbb{R}^2$ is closed, if and only if $\alpha = 0$.

3. **[Surface of revolution]** Let $I \subset \mathbb{R}$ be an interval and let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ be a smooth curve. Define the map $\psi : \mathbb{R}/2\pi\mathbb{Z} \times I \to \mathbb{R}^3$ by

$$\psi(s, t) := (\gamma_1(t)\cos(s), \gamma_1(t)\sin(s), \gamma_2(t)).$$

a) Draw the image of ψ, $\operatorname{im}(\psi) = S$ and give some examples of the surfaces of revolution.

b) When is ψ an immersion? When is it an embedding?

c) In the case when ψ is an embedding, calculate the metric g_{ij} from exercise 4 of exercise sheet 6.

d) In the case when ψ is an embedding, calculate the Christoffel symbols Γ_{ij}^k from exercise 5 of exercise sheet 6.

Solution:

a) The surface S is obtained by rotating the curve $(\gamma_1(t), \gamma_2(t)) \subset O_{yz}$ around $z-$axis. Well known surfaces of revolution are for example paraboloid, hyperboloid, ellipsoid, cylinder, torus.

b) The derivative of ψ is given by the matrix

$$\left(\frac{\partial \psi}{\partial s}, \frac{\partial \psi}{\partial t}\right) = \begin{pmatrix} -\gamma_1(t)\sin(s) & \dot{\gamma}_1(t)\cos(s) \\ \gamma_1(t)\cos(t) & \dot{\gamma}_1(t)\sin(s) \\ 0 & \dot{\gamma}_2(t) \end{pmatrix}.$$

Hence $d\psi$ is injective if and only if $\gamma_1(t) \neq 0$ and $\dot{\gamma}_2(t)^2 + \dot{\gamma}_1(t)^2 \neq 0$.

In order for ψ to be an embedding we need in addition that ψ is injective and proper. This holds true if and only if the curve $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ injective and proper.

c) The metric with respect to ψ has the components

$$g_{11}(s, t) = \left\| \frac{\partial \psi(s, t)}{\partial s} \right\| = \gamma_1(t)^2, \qquad g_{22}(s, t) = \left\| \frac{\partial \psi(s, t)}{\partial t} \right\| = \dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2$$

$$g_{12}(s, t) = g_{21}(s, t) = \left\langle \frac{\partial \psi(s, t)}{\partial s}, \frac{\partial \psi(s, t)}{\partial t} \right\rangle = 0$$

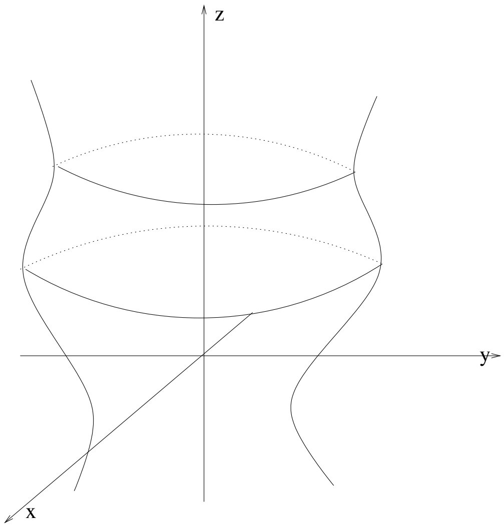

Figure 1: Surface of revolution

d) Since $g = (g_{ij})$ is a diagonal matrix, we can directly write down its inverse $g^{-1} = (g^{ij})$ and get

$$g^{11}(s,t) = \frac{1}{(\gamma_1(t))^2}, \quad g^{22}(s,t) = \frac{1}{\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2}, \quad g^{12}(s,t) = g^{21}(s,t) = 0.$$

We calculate first the symbols

$$\Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

(of course we must replace $x^1 = s$ and $x^2 = t$ for this to make sense). Note that $\Gamma_{kij} = \Gamma_{kji}$. When calculating these symbols, most terms vanish since g is a diagonal matrix and $\partial_s g = 0$. We obtain

$$\Gamma_{211}(s,t) = -\gamma_1(t)\dot\gamma_1(t), \qquad \Gamma_{112}(s,t) = \Gamma_{121}(s,t) = \dot\gamma_1(t)\gamma_1(t)$$

$$\Gamma_{222}(s,t) = \dot\gamma_1(t)\ddot\gamma_1(t) + \dot\gamma_2(t)\ddot\gamma_2(t)$$

and all the other symbols vanish. Moreover, since g is a diagonal matrix, the Christoffel symbols Γ^k_{ij} are given by $\Gamma^k_{ij} = g^{kk}\Gamma_{kij}$. This yields

$$\Gamma^2_{11}(s,t) = \frac{-\gamma_1(t)\dot\gamma_1(t)}{\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2}, \qquad \Gamma^1_{12}(s,t) = \Gamma^1_{21}(s,t) = \frac{\dot\gamma_1(t)}{\gamma_1(t)}$$

$$\Gamma^2_{22}(s,t) = \frac{\dot\gamma_1(t)\ddot\gamma_1(t) + \dot\gamma_2(t)\ddot\gamma_2(t)}{\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2}$$

4. For $p, q \in M$ define

$$\Omega_{p,q} := \Omega_{p,q}(M) := \{\gamma \in C^\infty([0,1], M) : \gamma(0) = p, \gamma(1) = q\}.$$

A map $\mathbb{R} \to \Omega_{p,q}$, $s \mapsto \gamma_s$ is called smooth path of curves, if the map joint map $\mathbb{R} \times [0,1] \to M$, $(s,t) \mapsto \gamma_s(t)$ is smooth as a function of s and t.

a) Let $\gamma : \mathbb{R} \to \Omega_{p,q}$ be a smooth path of curves. Prove that $X := \frac{d}{ds}\big|_{s=0}\gamma_s \in \mathrm{Vect}(\gamma_0)$ with $X(0) = X(1) = 0$.

b) Conversely, let $\gamma_0 \in \Omega_{p,q}$ and $X \in \mathrm{Vect}(\gamma_0)$ with $X(0) = X(1) = 0$ be given. Prove that there exists a smooth path of curves $\gamma : \mathbb{R} \to \Omega_{p,q}$, passing through γ_0 at $s = 0$, such that $X := \frac{d}{ds}\big|_{s=0} \gamma_s$.

This proves the identification

$$T_\gamma \Omega_{p,q} = \{X \in \mathrm{Vect}(\gamma) : \ X(0) = 0, \ X(1) = 0\}.$$

Solution: **Step 1:** First suppose that γ is contained in one coordinate chart. Let $U \subset M$ be an open set containing γ and let $\phi : U \xrightarrow{\cong} \Omega$, where $\Omega \subset \mathbb{R}^m$ is open. Then $a(t) = \phi(\gamma(t)) = \phi_*(\gamma), \ t \in [0,1]$ is a smooth curve in $\Omega \subset \mathbb{R}^m$. Let $\xi(t) = \phi_*(X(t)) = d\phi(\gamma(t))X(t)$. Now define $a_s(t) = a(t) + s\xi(t)$ and for $\epsilon > 0$ small enough we have $a_s(t) \subset \Omega, \ $ for $s \in (-\epsilon, \epsilon)$. Finally define $\gamma_s(t) := \phi^{-1}(a_s(t)) = \phi^*(a_s(t))$.
Step 2: In the case that γ isn't contained in a single chart cover it with finitely many charts $\phi_i : U_i \to \Omega = B_1(0) \subset \mathbb{R}^m, \ i = 1, \cdots, n$ and suppose that $\gamma|_{[t_i-1, t_i]} \subset U_i$ and construct on each segment curves $a^i_s : [t_{i-1}, t_i] \to \Omega, \ s \in (-\epsilon, \epsilon)$, as above, or equivalently the curves γ^i_s (we actually have that a curve a^i_s is defined on some interval $(t_{i-1} - \delta, t_i + \delta)$ for some small δ, with exception $i = 1, n$, for $n = 1$ we have the interval $[0, t_1 + \delta)$). The problem is how to connect these curves so that we get a smooth curve. This reduces to the problem how to connect two subsequent curves. Let γ^1_s be as in step 1, then the curve $\phi_2(\gamma^1_s) = b^2_s(t), \ t \in (t_1 - \delta, t_1 + \delta)$ is a smooth curve which satisfies $b^2_0(t) = \phi_2(\gamma(t)) = a^2(t)$, and it satisfies $\frac{d}{ds}\big|_{s=0} b^2_s(t) = \xi_2(t)$, where $\xi_2 = (\phi_2)_* X(t)$. Take a smooth cut-off function $\beta : \mathbb{R} \to [0,1]$ with the property

$$\beta(t) = \begin{cases} 1, t < t_1 \\ 0, t > t_1 + \delta \end{cases}$$

and define $c_s(t) = \beta(t) b^2_s(t) + (1 - \beta(t)) a^2_s(t)$, this is a smooth family of curves in $\mathbb{R}^m$, for small s it is contained in Ω. Notice that $c_0(t) = a^2(t) = \phi_2(\gamma(t))$ and that $\frac{d}{ds}|_{s=0} c_s(t) = \xi_2(t)$. Thus we get a smooth extension of γ^1_s by taking $\phi_2^{-1}(c_s(t))$. We precede analogously.

Another way you could solve this exercise is the following:

Define $Y_t(p) := \Pi(p)X(t)$ for $t \in I$, $p \in M$ and observe that $Y_t \in \mathrm{Vect}(M)$. Denote by Ψ_{Y_t} the flow of Y_t on M. Then there is $\epsilon > 0$ such that we can define $\gamma_s(t) := \Psi^s_{Y_t}(\gamma(t))$ for $s \in (-\epsilon, \epsilon)$ and $t \in [0,1]$. Thus we get $\gamma_0(t) = \gamma(t)$ and $\frac{d}{ds}\big|_{s=0} \gamma_s(t) = Y^t(\gamma(t)) = X(t)$.

5. Let $U \subset \mathbb{R}^m$ be an open set and $g = (g_{ij}) : U \to \mathbb{R}^{m \times m}$ be a smooth map with values in the space of positive definite symmetric matrices. Let $p, q \in U$ with $p \neq q$.

a) For a smooth function $L : U \times \mathbb{R}^m \to \mathbb{R}$ consider the functional

$$E : \Omega_{p,q}(U) \to \mathbb{R}, \qquad E(\gamma) := \int_0^1 L(\gamma(t), \dot\gamma(t))\, dt$$

on the space of paths $\Omega_{p,q}(U)$ in U. Seeing E as a function on $\Omega_{p,q}$, prove that critical points of E are exactly those paths that fulfill the Euler-Lagrange equations of this variational problem which have the form

$$\frac{d}{dt} \frac{\partial L}{\partial \xi^k}(\gamma(t), \dot\gamma(t)) = \frac{\partial L}{\partial x_k}(\gamma(t), \dot\gamma(t)), \quad k = 1, \cdots m, \quad t \in [0,1]. \qquad (1)$$

Here we denote $L = L(x, \xi)$ with $(x, \xi) \in U \times \mathbb{R}^m$.

b) Specialise to the case

$$L(x,\xi) := \frac{1}{2}\sum_{i,j=1}^m \xi^i g_{ij}(x)\xi^j.$$

In this case, we call E the energy functional. Prove that the Euler-Lagrange equations (1) are equivalent to the geodesic equations

$$\ddot\gamma^k(t) + \sum_{i,j=1}^m \Gamma^k_{ij}(\gamma(t))\,\dot\gamma^i(t)\dot\gamma^j(t) = 0, \quad k = 1,\cdots,m, \quad t\in[0,1]$$

where Γ^k_{ij} are Christoffel symbols.

Hint: For a), we consider $\Omega_{p,q}$ formally as a manifold and calculate for a tangent vector $X := \frac{d}{ds}\big|_{s=0}\gamma_s \in \mathrm{Vect}(\gamma)$ the derivative by $dE(\gamma)X = \frac{d}{ds}\big|_{s=0} E(\gamma_s)$.

Solution:

a) Take $X := \frac{d}{ds}\big|_{s=0}\gamma_s \in \mathrm{Vect}(\gamma)$, and calculate

$$\begin{aligned}
dE(\gamma)X &= \frac{d}{ds}\bigg|_{s=0}\int_0^1 L(\gamma_s(t),\dot\gamma_s(t))\,dt \\
&= \sum_{k=0}^m \int_0^1 \left(\frac{\partial}{\partial x^k}L(\gamma(t),\dot\gamma(t))\,\frac{d}{ds}\bigg|_{s=0}\gamma_s^k(t) + \frac{\partial}{\partial \xi^k}L(\gamma(t),\dot\gamma(t))\frac{d}{dt}\,\frac{d}{ds}\bigg|_{s=0}\gamma_s^k(t)\right) dt \\
&= \sum_{k=0}^m \int_0^1 \left(\frac{\partial}{\partial x^k}L(\gamma(t),\dot\gamma(t))X^k(t) - \frac{d}{dt}\left(\frac{\partial}{\partial \xi^k}L(\gamma(t),\dot\gamma(t))\right)X^k(t)\right) dt \\
&\qquad + \sum_{k=0}^m \left[\frac{\partial}{\partial \xi^k}L(\gamma(t),\dot\gamma(t))X^k(t)\right]_0^1 \\
&= \sum_{k=0}^m \int_0^1 \left(\frac{\partial}{\partial x^k}L(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\left(\frac{\partial}{\partial \xi^k}L(\gamma(t),\dot\gamma(t))\right)\right)X^k(t)\,dt,
\end{aligned}$$

where the second line uses integration by part and the last uses $X(0) = X(1) = 0$.

Now γ being a critical point for E means that $dE(\gamma)X = 0$ for all $X \in \mathrm{Vect}(\gamma)$. The equation

$$dE(\gamma)X = \sum_{k=0}^m \int_0^1 \left(\frac{\partial}{\partial x^k}L(\gamma(t),\dot\gamma(t)) - \frac{d}{dt}\left(\frac{\partial}{\partial \xi^k}L(\gamma(t),\dot\gamma(t))\right)\right)X^k(t)\,dt$$

directly implies that γ is a critical point whenever γ fulfills the Euler-Lagrange equations.

The converse follows from the following simple fact: If f is a continuous function and $\int_0^1 f(t)\varphi(t)\,dt = 0$ for all $\varphi \in C^\infty([0,1])$, then follows $f \equiv 0$. Using vector fields of the shape $X_{k,\varphi}(t) = \varphi(t)e_k$ where e_k is the k-th base vector of $\mathbb{R}^m$, it follows that any critical point γ satisfies the Euler-Lagrange equations.

b) To simplify the expression we shall use Einstein summation convention (when an index appears twice- once in an upper and once in a lower position we are summing over all its possible values). Thus, in this notation $L(x,\xi) = g_{ij}(x)\xi^i\xi^j$ and

$$\frac{\partial L}{\partial \xi^k}(x,\xi) = g_{ki}(x)\xi^i + g_{ik}(x)\xi^i = 2g_{ki}(x)\xi^i, \quad \text{as } g_{ik} = g_{ki}.$$

Hence,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \xi^k}(\gamma,\dot\gamma) &= \frac{d}{dt}[2g_{ki}(\gamma(t))\dot\gamma^i(t)] = \\ &= 2\partial_j g_{ki}(\gamma)\dot\gamma^j\dot\gamma^i + 2g_{ki}(\gamma)\ddot\gamma^i \\ &= \partial_j g_{ki}(\gamma)\dot\gamma^j\dot\gamma^i + \partial_i g_{kj}(\gamma)\dot\gamma^j\dot\gamma^i + 2g_{ki}(\gamma)\ddot\gamma^i \qquad (2)\end{aligned}$$

Similarly as,

$$\frac{\partial L}{\partial x^k}(x,\xi) = \partial_k g_{ij}(x)x^i x^k$$

it follows

$$\frac{\partial L}{\partial x^k}(\gamma,\dot\gamma) = \partial_k g_{ij}(\gamma)\dot\gamma^i\dot\gamma^j. \qquad (3)$$

Now identifying the equalities (2) and (3) we get

$$2g_{ki}\ddot\gamma^i(t) + (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})\dot\gamma^i\dot\gamma^j = 0.$$

Multiplying the previous inequality with the inverse matrix of the matrix $g^{-1} = (g^{\ell k})$ (and notice that $g^{\ell i}g_{ik} = \delta_{\ell k}$) we get

$$\ddot\gamma^\ell + \Gamma^\ell_{ij}\dot\gamma^i\dot\gamma^j = 0.$$

6. Let Arccos : $[-1,1] \to [0,\pi]$ denote the inverse of the function $\cos|_{[0,\pi]}$. Consider the function on S^2 given by

$$d(p,q) = \mathrm{Arccos}(\langle p,q\rangle), \qquad \text{for } p,q \in S^2.$$

a) Prove d is the intrinsic distance function on S^2, i.e. $d(p,q) = \inf L(\gamma)$ where the infimum is taken over all curves connecting p to q.

b) Prove that d induces the same topology as the subset topology on $S^2 \subset \mathbb{R}^3$.

Hint: In a), use spherical coordinates and take adventage of the symmetrie of the sphere. For b), you could prove equivalence to another distance function on S^2 which induces the subset topology.

Solution:

a) We work with spherical coordinates defined by

$$P : [0,2\pi]\times[0,\pi] \to S^2, \qquad P(\varphi,\theta) := (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)).$$

We need that the following properties which are easy to verify:

$$\langle \partial_\theta P(\varphi,\theta), \partial_\varphi P(\varphi,\theta)\rangle = 0, \qquad ||\partial_\theta P(\varphi,\theta)|| = 1.$$

This says geometrically, that the two coordinate vectors are orthogonal at each point and that $\partial_\theta P(\varphi,\theta)$ has unit length.

For any rotation $A \in \mathrm{SO}(3)$ of the sphere, we have $\langle Ap, Aq\rangle = \langle p,q\rangle$ and $d(p,q) = d(Ap,Aq)$ for the intrinsic distance, as the rotation preserves the length any paths. We may hence assume that $p = (0,0,1)$ is the north pole and $q = (\sin(\theta_0), 0, \cos(\theta_0))$ for some $\theta_0 \in [0,\pi]$. Let $\gamma : [0,1] \to S^2$ be any smooth path satisfying $\gamma(0) = p$ and

$\gamma(1) = q$. Write $\gamma(t) = P(\varphi(t), \theta(t))$. By the properties of P mentioned above, it follows that

$$\begin{aligned} ||\dot{\gamma}(t)||^2 &= ||\partial_\theta P(\varphi(t), \theta(t))\dot{\theta}(t)||^2 + ||\partial_\varphi P(\varphi(t), \theta(t))\dot{\varphi}(t)||^2 \\ &\geq ||\partial_\theta P(\varphi(t), \theta(t))\dot{\theta}(t)||^2 \\ &= |\dot{\theta}(t)|^2 \end{aligned}$$

Hence

$$L(\gamma) := \int_0^1 ||\dot{\gamma}(t)||\, dt \geq \int_0^1 |\dot{\theta}(t)|\, dt \geq \int_0^1 \dot{\theta}(t)\, dt = \theta_0.$$

On the other hand $\langle p, q\rangle = \cos(\theta_0)$. We have thus shown $L(\gamma) \geq \operatorname{Arccos}(\langle p, q\rangle)$ and hence $d(p, q) \geq \operatorname{Arccos}(\langle p, q\rangle)$. To see that equality holds, simply consider the curve along the great circle from p to q

$$\tilde{\gamma} : [0, 1] \to S^2, \qquad \tilde{\gamma}(t) = (\sin(t\theta_0), 0, \cos(1 + t(\theta_0 - 1)))\,.$$

For this curve, all inequalities encountered above, become equalities and in particular $L(\tilde{\gamma}) = \theta_0 = \operatorname{Arccos}(\langle p, q\rangle)$.

b) Denote by $d_0(p, q) := ||p - q||$ the metric on S^2 induced by the Euclidean distance in the ambient space $\mathbb{R}^3$. By exercise 5 on exercise sheet 1, it follows that this metric induces the same topology as the subspace topology. We show in the following that the metric d_0 is equivalent to the intrinsic metric d. From this follows that both metrics induce the same topology on S^2 and establishes the claim.

We can write the metric $d_0(p, q)$ alternatively as

$$d_0(p, q) = \sqrt{\langle p - q, p - q\rangle} = \sqrt{|p|^2 + |q|^2 - 2\langle p, q\rangle} = \sqrt{2 - 2\langle p, q\rangle}.$$

As a side note, we clearly have $d_0(p, q) \leq d(p, q)$, as the straight line connecting p to q has minimal length in $\mathbb{R}^3$. On the other hand

$$\frac{d_0(p, q)}{d(p, q)} = \frac{\sqrt{2 - 2\langle p, q\rangle}}{\operatorname{Arccos}(\langle p, q\rangle)} = \sqrt{\frac{2 - 2\cos(\theta)}{\theta^2}} = \frac{2\sin(\theta/2)}{\theta}$$

where $\theta := \operatorname{Arccos}(\langle p, q\rangle) \in [0, \pi]$. The right hand side is a continuous function of θ on a compact interval $[0, \pi]$ (extended by 1 at $\theta = 0$) and a careful analysis shows that its minimum is 1 and its maximum is $\frac{2}{\pi}$. Thus

$$d_0(p, q) \leq d(p, q) \leq \frac{2}{\pi} d_0(p, q).$$