Solution 8

1. Let I be an interval and $\gamma \in C^\infty(I, S^n)$. Show that the following are equivalent:

(i) γ is a geodesic.

(ii) $|\dot\gamma|$ is constant and $\mathrm{im}(\gamma) \subset S^n$ is a big circle, i.e. there exists a 2−plane $P \subset \mathbb{R}^{n+1}$ through the origin such that $\mathrm{im}(\gamma) \subset P \cap S^n$.

Hint: To calculate the exponential map, use the second fundamental form $h_p(v,w) = -p\langle v, w\rangle$ calculated in exercise 3 of sheet number 5.

Solution: We can write down the geodesic equation using Exercise 3 of sheet number 5. There we calculated the second fundamental form as $h_p(v,w) = -p\langle v,w\rangle$. Thus we see $\gamma : \mathbb{R} \to S^n$ to be a geodesic of velocity $c > 0$ if it satisfies

$$\ddot\gamma(t) = -\gamma(t)\langle\dot\gamma(t), \dot\gamma(t)\rangle = -c^2\gamma(t).$$

So γ is of the form $\gamma(t) = a\cos(ct) + b\sin(ct)$ for some $a, b \in \mathbb{R}^{n+1}$. Requiring $\gamma(0) = p$ and $\dot\gamma(0) = v$ with $|v| = c$, we get for $p \in S^n$ and $v \in T_pS^n$

$$\exp_p(v) = p\cos(|v|) + \frac{v}{|v|}\sin(|v|).$$

As a sanity test, we get from $v \perp p$ that $|\exp_p(v)| = 1$ for all $v \in T_pS^n$.

Next given a 2-plane P going through 0. This can be written (in a non unique way) as $P = \mathrm{span}(p, v)$ for some $p \in S^n$ and $0 \neq v \in T_pS^n$. Thus all points of $P \cap S^n$ are given by $q = \lambda p + \mu\frac{v}{|v|}$ for some $\lambda, \mu \in \mathbb{R}$ with $|q| = 1$. However, as $p \perp v$, we get $\lambda^2 + \mu^2 = |q|^2 = 1$, i.e. $P \cap S^n = \mathrm{im}(t \mapsto \exp_p(tv))$. This proves the equivalence as any geodesic is of the form $t \mapsto \exp_p(tv)$ for some $v \in T_pS^n$.could use vector product and prove that $\gamma(t) \times \dot\gamma(t) = \mathrm{const.}$

2. Let ψ be a parametrisation of the surface of revolution S, as in exercise 3 of sheet number 7. The image of ψ of the curves $s = \mathrm{const.}$ and $t = \mathrm{const.}$ are called *meridians* and *parallels*. Suppose that the curve γ in the definition of ψ is parametrised by arc length i.e. $|\dot\gamma| = 1$. Prove and discuss the following:

a) The meridians, i.e. the curves $\psi_s(t) : I \to \mathbb{R}^3$, $\psi_s(t) = \psi(s,t)$ are geodesics on S.

b) When is a parallel $\psi^t : \mathbb{R} \to \mathbb{R}^3$, $\psi^t(s) := \psi(s,t)$ a geodesic?

c) Describe the geodesics on the standard cylinder in $\mathbb{R}^3$,

$$Z := \{(x,y,z) \in \mathbb{R}^3 \,|x^2 + y^2 = 1\}.$$

Solution:

a) Let us use the Christoffel symbols, we calculated in sheet number 7. Namely, the only non-zero symbols are

$$\Gamma^2_{11}(s,t) = -\gamma_1(t)\dot\gamma_1(t), \qquad \Gamma^1_{12}(s,t) = \Gamma^1_{21}(s,t) = \frac{\dot\gamma_1(t)}{\gamma_1(t)}$$
$$\Gamma^2_{22}(s,t) = \dot\gamma_1(t)\ddot\gamma_1(t) + \dot\gamma_2(t)\ddot\gamma_2(t).$$

The geodesic equation in local coordintates says, that the curve $\psi \circ c$ is a geodesic if and only if

$$\ddot c^k(\tau) + \sum_{i,j=1}^{2}\Gamma^k_{ij}(c(\tau))\dot c^i(\tau)\dot c^j(\tau) = 0, \qquad \text{for } k = 1, 2..$$

(here we denote $s = x^1$ and $t = x^2$ for the index notation).

For $c(\tau) = (s, \tau)$, we have $\dot{c}(\tau) = (0, 1)$ and $\ddot{c}(\tau) = (0, 0)$. Since $\dot{c}^1(\tau) = 0$, most terms in the geodesic equation vanishes: For $k = 1$ everything vanishies (including $\ddot{c}^1(\tau) = 0$). The equation for $k = 2$ simplifies to:

$$\dot{\gamma}_1(\tau)\ddot{\gamma}_1(\tau) + \dot{\gamma}_2(\tau)\ddot{\gamma}_2(\tau) = 0.$$

The left hand side is the derivative of $\frac{1}{2}(\dot{\gamma}_1(\tau)^2 + \dot{\gamma}_2^2(\tau))$ which is assumed to be constant. Thus the second geodesic equation is satisfied and the meridians are always geodesics.

b) We consider the curve $c(\tau) = (\tau, t)$. Thus we get $\dot{c}(\tau) = (1, 0)$ and $\ddot{c}(\tau) = (0, 0)$. In particular, $\dot{c}^2(\tau) = 0$ and in the geodesic equation for $k = 1$ everything vanishes. The equation for $k = 2$ simplifies to

$$\gamma_1(t)\dot{\gamma}_1(t) = 0.$$

As our surface of revolution is embedded, we have $\gamma_1(t) \neq 0$. Therefore, a parallel is a geodesic for $t \in I$ if and only if $\dot{\gamma}_1(t) = 0$.

c) We have that $\gamma(t) = (1, t)$ for all $t \in \mathbb{R}$ gives us the cylinder as surface of revolution. Therefore, we get the geodesic equations for the curve $\psi \circ c$ given by

$$\ddot{c}^1 = 0, \qquad \ddot{c}^2 = 0.$$

Thus all solutions are of the form $c(\tau) = (a\tau + b, c\tau + d)$ for $\tau \in \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$.

3. Let $p_0 \in M^m$. We know from the lectures that there exists $\epsilon > 0$ and a neighbourhood $U_\epsilon \subset M$ of the point p_0 such that

$$\exp_{p_0} : \{v \in T_{p_0}M : \ |v| < \epsilon\} \to U_\epsilon$$

is a diffeomorphism. Let $\{e_i\}_{i=1,\cdots m}$ be an orthonormal basis of $T_{p_0}M$. We define a diffeomorphism $\psi : B_\epsilon(0) \subset \mathbb{R}^m \to U_\epsilon$ by

$$\psi(x) := \exp_{p_0}\Big(\sum_{i=1}^m x^i e_i\Big).$$

a) Show that $g_{ij}(0) = \delta_{ij}$, for $i, j = 1, \cdots m$.

b) Show that $\Gamma_{ij}^k(0) = 0$ for $i, j, k = 1, \cdots m$.

These coordinates are called *normal coordinates centered at p_0*.

Solution:

a) We know from the lecture that $d\exp_{p_0}(0) : T_{p_0}M \to T_{p_0}M$ is the identity. Since

$$e : \mathbb{R}^m \to T_{p_0}M, \qquad e(x) = \sum_{i=1}^m x^k e_k$$

is a linear frame, the chain rule yields:

$$d\psi(0)x = \sum_{i=1}^m x^i e_i$$

Therefore,

$$g_{ij}(0) = \left\langle \frac{\partial\psi(0)}{\partial x^i}, \frac{\partial\psi(0)}{\partial x^j} \right\rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

b) Let $v \in \mathbb{R}^m$ be a unit vector and consider the curve $c : (-\epsilon, \epsilon) \to \mathbb{R}^m$, $c(t) := tv$. By definition of the exponential map, $\gamma : (-\epsilon, \epsilon) \to M$, $\gamma(t) := \psi(c(t)) = \exp_{p_0}(tv)$ is a geodesic and hence $c(t)$ satisfies the geodesic equations

$$\ddot{c}^k(t) + \sum_{i,j=1}^{m} \Gamma_{ij}^k(c(t))\dot{c}^i(t)\dot{c}^j(t) = 0.$$

At time $t = 0$ this equality gives us

$$\sum_{i,j=1}^{m} \Gamma_{ij}^k(0)v^i v^j = v^T A^k v = 0, \tag{1}$$

where $A^k = (\Gamma_{ij}^k(0))_{i,j=1}^m$ is a symmetric matrix. Since equality (1) must hold for any unit vector v, this implies that $A^k = 0$. One way to sees this is through the identity

$$(Av + v)^\top A(Av + v) - (Av)^\top A(Av) - v^\top Av = 2(Av)^\top Av = 2||Av||^2.$$

for a symmetric matrix A. Thus all Christoffel symbols vanish at 0.

4. Determine the radius of injectivity at all points of the following manifolds.

a) $T^2 := S^1 \times S^1$. **b)** $\mathbb{R}^2 \setminus \{0\}$. **c)** S^n.

Hint: For c), use the exponential map which you should already have calculated in exercise 1.

Solution:

a) We consider points on the torus to be given by two angles (θ, φ) modulo 2π using the map

$$\psi : \mathbb{R}^2 \to T^2, \qquad \psi(\theta, \varphi) := (e^{i\theta}, e^{i\varphi}).$$

The map ψ is 2π-periodic in both variables and for open sets $U \subset \mathbb{R}^2$ with diameter smaller then 2π, the restriction $\psi|_U : U \to \psi(U) \subset T^2$ is an honest chart. For any such chart we have

$$g_{ij}(\theta, \varphi) = \delta_{ij}$$

The formula for the Christoffel symbols in Exercise 5 of Exercise Sheet 6 then implies $\Gamma_{ij}^k(\theta, \varphi) = 0$. The geodesic equation for a curve $c(t) = (\theta(t), \varphi(t))$ are thus

$$\ddot{\theta}(t) = 0, \qquad \ddot{\varphi}(t) = 0.$$

In other words, $\psi(c(t))$ is a geodesic in T^2, if and only if $c(t)$ is a straight line in $\mathbb{R}^2$. From this one obtains that the exponential map is given by

$$\exp_{\psi(\theta_0,\varphi_0)} : \mathbb{R}^2 \to T, \qquad \exp_{\psi(\theta_0,\varphi_0)}(x^1, x^2) = (e^{\mathbf{i}(\theta_0 + x^1)}, e^{\mathbf{i}(\varphi_0 + x^2)}).$$

The restriction $\exp|_{B_{2\pi}(0)}$ is clearly injective. Moreover, the differential

$$d\exp_{\psi(\theta_0,\varphi_0)}(x^1, x^2)[\xi^1, \xi^2] = (\mathbf{i}e^{\mathbf{i}(\theta_0 + x^1)}\xi^1, \mathbf{i}e^{\mathbf{i}(\varphi_0 + x^2)}\xi^2)$$

is always an isomorphism and thus $\exp|_{B_\pi(0)}$ is a smooth injective map which by inverse function theorem is locally diffeomorphic. Thereby the injectivity radius is at least π. But it can cannot be larger, since one has $\exp_{\psi(\theta_0,\varphi_0)}(0, \pi) = \exp_{\psi(\theta_0,\varphi_0)}(0, -\pi)$. This proves that the injectivity radius is 2π .

b) We see that geodesics are straight lines, and that this is not a complete manifold, as any ray $t \mapsto tv$ for $t \neq 0$ and $v \neq 0$ is a geodesic which cannot be extended. Thus the injectivity radius of (x, y) is given by $|(x, y)|$.

c) For S^n, we get the exponential map

$$\exp_p(v) = \cos(|v|)p + \sin(|v|)\frac{v}{|v|}.$$

So assume $\exp_p(v) = \exp_p(w)$. Then as $\{p, v\}$ and $\{p, w\}$ are linearly independent, we must have $\cos(|v|) = \cos(|w|)$ and $\frac{v}{|v|}\sin(|v|) = \frac{w}{|w|}\sin(|w|)$. Thus we have either that $|v| = |w| = k\pi$ for $k \in \mathbb{Z}$ or w and v are collinear. In the collinear case, we further get that either $v = w$ or $\frac{v}{|v|} = -\frac{w}{|w|}$ and in the latter case, this then further gives for $|v| + |w| = 2k\pi$ for $k \in \mathbb{Z}$. Thus the biggest ball on which $\exp_p$ is injective is $B_\pi(0)$. To see that π is indeed the injectivity radius, we need to check that the differential $d\exp_p(v)$ is injective for all $v \in B_\pi(0)$. Thus let us calculate for $v \neq 0$

$$d\exp_p(v)\hat{v} = -p\sin(|v|)\frac{\langle \hat{v}, v\rangle}{|v|} + \frac{v}{|v|}\cos(|v|)\frac{\langle \hat{v}, v\rangle}{|v|} + \frac{\hat{v}}{|v|}\sin(|v|) - \frac{v\langle \hat{v}, v\rangle}{|v|^3}\sin(|v|).$$

Thus we have that $d\exp_p(v)v = -p\sin(|v|)|v| + v\cos(|v|)$ and $d\exp_p(v)w = w\sin(|v|)$ for any vector $w \in T_pS^n$ which has $w \perp v$. So as long as $|v| < \pi$, $d\exp_p(v)$ is injective.

5. Find an example of a manifold M such that any two points $p, q \in M$ can be connected by a geodesic. Find an example of a manifold that does not have this property.

Solution: We have seen some examples in this exercise sheet. For example on S^n, T^2 or the cylinder, any two points are connected by a geodesic, whereas on $\mathbb{R}^2 \setminus \{0\}$, the points $(1, 0)$ and $(-1, 0)$ are not connected by any geodesic.

6. Recall from Exercise Sheet 1, that $O(n)$ is a manifold with tangent spaces

$$T_gO(n) = \{\xi \in \mathbb{R}^{n\times n} : g^\top \xi + \xi^\top g = 0\} \quad \text{for } g \in O(n).$$

a) The euclidean inner product on $\mathbb{R}^{n\times n}$ is defined by

$$\langle \xi, \eta\rangle := \mathrm{tr}(\xi^\top \eta).$$

Verify the following formulae.

(i) $\Pi(g)\xi = \frac{1}{2}(\xi - g\xi^\top g)$ for all $g \in O(n)$ and $\xi \in \mathbb{R}^{n\times n}$.,

(ii) $h_g(\xi, \eta) = -\frac{1}{2}g(\xi^\top \eta + \eta^\top \xi)$ for all $g \in O(n)$ and $\xi, \eta \in T_gO(n)$.

b) Recall the matrix exponential given by $\exp(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$. Prove that the geodesic γ starting at $g \in O(n)$ with initial velocity $\dot{\gamma}(0) = \xi \in T_gO(n)$ is given by

$$\gamma(t) = g\exp(tg^\top \xi) \quad \text{for } t \in \mathbb{R}. \qquad (2)$$

In other words, the (geodesic) exponential map at the identity $\exp_{\mathbb{1}}$ is the (matrix) exponential map.

c) Let $G \subset O(n)$ be a compact Lie subgroup. Prove that (2) still holds for all $g \in G$ and $\xi \in T_gG$.

(A theorem from Lie theory says that closed subgroups of a Lie group are also submanifolds.)

d) Determine

$$r_0 := \inf\{||\xi|| \,:\, 0 \neq \xi \in \exp_{\mathbb{1}}^{-1}(\mathbb{1})\}.$$

(It is a convincing and true fact that the injectivity radius of $O(n)$ is given by $\frac{r_0}{2}$. However, the proof is quite sophisticated.)

Hint: In b) use the Gauss-Weingarten formula to deduce that $\dot\gamma(t)^\top\dot\gamma(t)$ is constant and then verify that $g\exp(tg^\top\xi)$ satisfies the same ODE as $\gamma(t)$. For c) recall that in a small neighbourhood of a point, geodesics are characterised as being length minimising curves. For d), use a base change to get a skew-symmetric matrix into its normal form with diagonal blocks $\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix}$ or $\begin{pmatrix}0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

Solution:

a) (i) The tangent space $T_gO(n)$ contains all ξ which are anti-symmetric with respect to the inner product defined by g. We claim that its orthogonal complement consists of all ξ which are symmetric with respect to g:

$$T_g^\perp O(n) = \mathrm{Sym}(g) = \{\xi\in\mathbb{R}^{n\times n} : g^\top\xi - \xi^\top g = 0\}.$$

Indeed, we have for $\xi\in T_gO(n)$ and $\eta\in\mathrm{Sym}(g)$

$$\langle\xi,\eta\rangle = \mathrm{tr}(\xi^\top\eta) = \mathrm{tr}(\xi^\top g^\top g\eta) = -\mathrm{tr}(g\xi\eta^\top g^\top) = -\mathrm{tr}(\xi\eta^\top) = -\langle\xi,\eta\rangle$$

which implies $\langle\xi,\eta\rangle = 0$. Furthermore, an arbitrary matrix $\xi\in\mathbb{R}^{n\times n}$ decomposes into a symmetric and anti symmetric matrix (with respect to g) by the formula

$$\xi = \frac{1}{2}(\xi - g\xi^\top g) + \frac{1}{2}(\xi + g\xi^\top g)\in T_gO(n)\oplus\mathrm{Sym}(g).$$

This establishes the claim $T_g^\perp O(n) = \mathrm{Sym}(g)$ and directly implies the formula for $\Pi(g)$.

(ii) The derivative of $\Pi(g)\xi = \frac{1}{2}(\xi - g\xi^\top g)$ is given by the product rule as

$$d(\Pi(g)\xi)\hat g = \frac{1}{2}(-\hat g\xi^\top g - g\xi^\top\hat g) = -\frac{1}{2}g(\hat g^\top\xi + \xi^\top\hat g)$$

The last step uses $\xi^\top,\hat g\in T_gO(n)$ which yields $\hat g\xi^\top g = \hat g g^\top\xi = g\hat g^\top\xi$. Recall that by linearity of the differential, $d(\Pi(g)\xi)\hat g = (d\Pi(g)\hat g)\xi$. We thus get for the second fundamental form the formula:

$$h_g(\xi,\eta) = -\frac{1}{2}g(\xi^\top\eta + \eta^\top\xi)$$

b) It follows from the Gauss-Weingarten formula applied to the vector field $\dot\gamma\in\mathrm{Vect}(\gamma)$ that

$$\ddot\gamma(t) = h_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))$$

if and only if γ satisfies the geodesic equation $\nabla\dot\gamma = 0$. The formula for the second fundamental form from above yields

$$\ddot\gamma(t) = -\gamma(t)\dot\gamma(t)\dot\gamma^\top(t)$$

Multiplying this equation by $\gamma(t)^\top$ yields

$$\gamma(t)^\top\ddot\gamma(t) + \dot\gamma(t)\dot\gamma^\top(t) = 0.$$

The left hand side is the derivative of $\gamma(t)^\top\dot\gamma(t)$ and hence $\gamma(t)^\top\dot\gamma(t) = \gamma(t)^{-1}\dot\gamma(t)\equiv\xi_0\in T_{\mathbb{1}}O(n)$ is constant. Evaluating this at $t=0$ yields $\xi_0 = \gamma(0)^T\dot\gamma(0) = g^T\xi$.

We can summarize our findings as follows: Define $X \in \mathrm{Vect}(O(n))$ by $X(h) = h\xi_0$. Then $\gamma(t)$ is the unique solution of the equation

$$\gamma(0) = g, \qquad \dot{\gamma}(t) = X(\gamma(t))$$

(which is given by the flow of X). Now define $\tilde{\gamma}(t) := g\exp(tg^\top \xi) = g\exp(t\xi_0)$. We clearly have $\tilde{\gamma}(0) = g = \gamma(0)$ and to conclude $\tilde{\gamma}(t) = \gamma(t)$ is suffices to verify that $\tilde{\gamma}(t)$ is a flow line of X:

$$\begin{aligned}\dot{\tilde{\gamma}}(t) &= \frac{d}{dt}\left(g\sum_{k=0}^{\infty}\frac{(t\xi_0)^k}{k!}\right) = g\sum_{k=1}^{\infty}\frac{kt^{k-1}\xi_0^k}{k!} = g\left(\sum_{k=1}^{\infty}\frac{t^{k-1}\xi_0^{k-1}}{(k-1)!}\right)\xi_0 \\ &= g\exp(t\xi_0)\xi_0 = X(\tilde{\gamma}(t))\end{aligned}$$

c) Let $G \subset O(n)$ be a compact Lie subgroup. For $g \in G$ and $\xi \in T_gG$, the argument above shows that the geodesic $\gamma(t) = g\exp(tg^\top \xi)$ remains in the subgroup G. (It is a flow line of the vector field $X \in \mathrm{Vect}(G)$ with $X(h) = h(g^{-1}\xi)$). Since $\gamma(t)$ is a geodesic in $O(n)$, it is locally length minimizing in $O(n)$. But then it is also locally length minimizing in G and hence a geodesic in G.

d) We first recall a fact from linear algebra. Let $\xi \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix satisfying $\xi^\top = -\xi$. Then there exists a matrix $T \in O(n)$ and $\lambda_1, \dots \lambda_r \in \mathbb{R}$ such that

$$T^{-1}\xi T = \begin{pmatrix} 0 & -\lambda_1 & & & & & & \\ \lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & -\lambda_r & & & \\ & & & \lambda_r & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}$$

We briefly indicate how this result is obtained from standard results: Since $\xi = -\xi^T$ is skew-symmetric, it is in particular a normal operator and hence diagonalizable. Since ξ is a real matrix and skew-hermitian, the non-zero eigenvalues are purely immaginary and come in complex conjugated pairs $\mathbf{i}\lambda_1, -\mathbf{i}\lambda_1, \dots, \mathbf{i}\lambda_r, -\mathbf{i}\lambda_r$ (repeated according to their multiplicity). Moreover, if v is an eigenvector for $\mathbf{i}\lambda_j$, then $\bar{v}$ is an eigenvector for $-\mathbf{i}\lambda_j$. We may thus choose a unitary basis $v_1, \bar{v}_1, \dots, v_r, \bar{v}_r$ for these eigenspaces. Define

$$f_j = \frac{v_j + \bar{v}_j}{||v_j + \bar{v}_j||}, \qquad e_j = \frac{v_j - \bar{v}_j}{\mathbf{i}||v_j - \bar{v}_j||}$$

Using that v_j and $\bar{v}_j$ are orthogonal vectors, it follows that e_j, f_j are well-defined, $||v_j - \bar{v}_j|| = ||v_j + \bar{v}_j||$, and $\xi f_j = -\lambda_j e_j$ and $\xi e_j = \lambda_j f_j$. Now complete $e_1, f_1, \dots, e_r, f_r$ to an orthonormal basis of $\mathbb{R}^n$ and note that ξ has the desired shape in this basis.

We show next that

$$2\sqrt{2}\pi = r_0 := \min\{||\xi|| \,:\, 0 \neq \xi \in \exp^{-1}(\mathbb{1})\}$$

Let $\xi \in T_{\mathbb{1}}O(n)$ be given and let $T \in O(n)$ be such that $T^{-1}\xi T = T^\top \xi T$ has the normal form discussed above. Then

$$\exp(T^{-1}\xi T) = \begin{pmatrix} \cos(\lambda_1) & -\sin(\lambda_1) & & & & & & \\ \sin(\lambda_1) & \cos(\lambda_1) & & & & & & \\ & & \ddots & & & & & \\ & & & \cos(\lambda_r) & -\sin(\lambda_r) & & & \\ & & & \sin(\lambda_r) & \cos(\lambda_r) & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

and

$$\exp(\xi) = T\exp(T^{-1}\xi T)T^{-1} = \mathbb{1} \qquad \Longleftrightarrow \qquad \exp(T^{-1}\xi T) = \mathbb{1}$$

Now suppose $||\xi|| < 2\sqrt{2}\pi$. Then

$$||\xi|^2 = ||T^\top \xi T||^2 = 2\lambda_1^2 + 2\lambda_2^2 + \cdots + 2\lambda_r^2$$

shows that $|\lambda_j| < 2\pi$ for all j and thus $\exp(T^{-1}\xi T) \neq \mathbb{1}$ which is equivalent to $\exp(\xi) \neq \mathbb{1}$. Conversely, consider the matrix

$$\xi_0 := \begin{pmatrix} 0 & -2\pi & & & \\ 2\pi & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

which satisfies $||\xi_0|| = 2\sqrt{2}\pi$ and $\exp(\xi_0) = \mathbb{1}$. This completes the proof of $r_0 = 2\sqrt{2}\pi$.