Solution 9

1. Let (M, d) be a metric space. Prove that the following are equivalent:

(i) Every closed and bounded subset $A \subset M$ is compact.

(ii) Every bounded sequence in M has a convergent subsequence.

Solution: We prove that $(i) \Rightarrow (ii)$.

Let $\{p_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Take $A = \overline{\{p_n\}_{n\in\mathbb{N}}}$. Then the set A is closed and bounded and by (i) it is compact. Thus, as A is sequentially compact the sequence $\{p_n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

We prove that $(ii) \Rightarrow (i)$.

Any sequence of points $\{q_n\}_{n\in\mathbb{N}},\ q_n \in A,\ n \in \mathbb{N}$ is bounded and hence has a convergent subsequence. The limit point q has to be an element of A as A is closed. Thus A is compact (sequentially compact).

2. Let $M \subset \mathbb{R}^k$ be a manifold and d the intrinsic distance function on M.

a) Assume M is closed as a subset of $\mathbb{R}^k$ and prove that (M, d) is complete.

b) Give an example of a manifold $M \subset \mathbb{R}^k$ which is complete but not closed.

c) Prove that if $M \subset \mathbb{R}^k$ is compact then M is geodesically complete.

Solution:

a) Let $\{q_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in M. By definition this means that for all $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that if $m, n \geq n_0$ we have $d(q_n, q_m) < \epsilon$. But as, $|q_n - q_m| \leq d(q_n, q_m)$, we have that $\{q_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\mathbb{R}^k$. As $\mathbb{R}^k$ is complete this sequence is also convergent. Thus there is a limit $q = \lim_{n\to\infty} q_n$, but $q \in M$ as M is closed.

b) Take $M = \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2 \mid x \in (0, +\infty)\}$. Then $\overline{M} = M \cup \{(0, t) :\ t \in [0, 1]\}$, but it is complete, as it is geodesically complete.

c) From $a)$ it follows that (M, d) is complete and from the theorem on completeness (4.5.5), it follows that M is geodesically complete.

3. Let $M \subset \mathbb{R}^k$ be a manifold, $I \subset \mathbb{R}$ an open interval, and $\gamma : I \to M$ a geodesic. Fix a point $t_0 \in I$. Show that there exists $\epsilon > 0$ such that for any two real numbers $t_0 - \epsilon < s < t < t_0 + \epsilon$ the restriction of γ to the interval $[s, t]$ is length minimizing, i.e. $L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$.

How large can you choose ϵ in the case $M = S^2$?

Hint: Define $\epsilon := \frac{1}{2} \inf\{\mathrm{inj}(y) : d(y, \gamma(t_0)) \leq r\}$ and use Theorem 4.4.4 to show that this does the job when r is sufficiently small.

Solution: Let $0 < r < \mathrm{inj}(\gamma(t_0))$ and define $0 < \epsilon := \frac{1}{2} \mathrm{inj}(\overline{U_r}(\gamma(t_0)))$, where

$$U_r(x) = \{y \in M : d(x, y) < r\},$$

is a so called geodesic ball. Let $t_0 - \epsilon < s < t < t_0 + \epsilon$ and let $p = \gamma(s),\ q = \gamma(t)$. As $d(p, q) < 2\epsilon \leq \mathrm{inj}(p)$ and as geodesic segments minimise the distance on $U_{\mathrm{inj}(p)}(p)$ (this follows from theorem 4.4.4), it follows that $L(\gamma|_{[s,t]}) = d(p, q)$.

The constant ϵ can be chosen to be maximally $\frac{\pi}{2}$ on the sphere.

4. **a)** How large can you choose the constant $r(p) > 0$ for $p \in M$ such that $U_{r(p)} := \{q \in M \mid d(p,q) < r(p)\}$ is geodesically convex in the cases

(i) $M = T^2$, (ii) $M = \mathbb{R}^2 \setminus \{0\}$, (iii) $M = S^2$.

b) Find a geodesically convex set U in a manifold M and points $p, q \in U$ such that the unique geodesic $\gamma : [0,1] \to U$ with $\gamma(0) = p$ and $\gamma(1) = q$ has length $L(\gamma) > d(p,q)$.

c) Find a set U in a manifold M such that any two points in U can be connected by a minimal geodesic in U, but U is not geodesically convex.

Hint: For b), take M to be the upper hemisphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z > 0\}$ together with the disc $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\}$, but smooth the corners along the circle $x^2 + y^2 = 1$, $z = 0$. Take U to be a large metric ball for the intrinsic distance in the upper hemisphere.

Solution:

a) (i) For T^2, $r(p)$ is equal to the injectivity radius π at any point $p \in T^2$. Geometrically, this is easily seen: Use the parametrization

$$\psi : \mathbb{R}^2 \to T^2, (\varphi, \theta) \mapsto (e^{\mathbf{i}\varphi}, e^{\mathbf{i}\theta})$$

from Exercise 4 on Exercise Sheet 8. We have calculated the exponential map and geodesics of the torus using this parametrization: For $p \in T^2$, choose any preimage $(\varphi_0, \theta_0) \in \mathbb{R}^2$ with $\psi(\varphi_0, \theta_0) = p$. Then ψ restricts to a diffeomorphism

$$\psi|_{B_\pi(\varphi_0,\theta_0)} : B_\pi(\varphi_0, \theta_0) \to U_\pi(p)$$

and straight lines in $B_\pi(\varphi_0, \theta_0)$ are maped to geodesics in $U_\pi(p)$ and visa versa. Since any two points in $B_\pi(\varphi_0, \theta_0)$ can be connected by a unique line, it follows that $U_\pi(p)$ is geodesically convex. For $r > \pi$, the neighborhood $U_r(p)$ contains at least two closed circles and cannot be geodesically convex.

(ii) For $\mathbb{R}^2 \setminus \{0\}$ we have by the same reasoning as above that $r(x,y)$ is equal to the injectivity radius $\mathrm{inj}(x,y) = |(x,y)|$.

(iii) For S^2, it will turn out to be half the injectivity radius. Due to the symmetries of the sphere it is enough to restrict ourselves to the north pole $p = p_N$. It follows from Exercise 6 in Exercise Sheet 7 that $U_r(p_N) = \{(x,y,z) \in S^2 : z > \cos(r)\}$. First, we see that if there is a great circle entirely contained in $U_r(p_N)$, then we have two distinct ways of connecting any two points on this great circle by geodesics. This happens as soon as $r > \frac{\pi}{2}$, as we then have the equator in $U_r(p_N)$. On the other hand if $r \leq \frac{\pi}{2}$, then for every $q \in U_r(p_N)$, $-q \notin U_r(p_N)$. Therefore, for any two points $q_1, q_2 \in U_r(p_N)$, the two-plane $E := \mathrm{span}\{q_1, q_2\}$ will define a great circle $E \cap S^2$ not entirely contained in $U_r(p_N)$ which defines a geodesic segment connecting q_1 and q_2 and this geodesic is also unique. Thus $r(p_N) = \frac{\pi}{2}$.

b) Fix $0 < \epsilon < \frac{\pi - 2}{2}$ small. Take $\gamma : I := [0, \frac{\pi}{2} + 1 - 2\epsilon + \ell(\epsilon)] \to \mathbb{R}^2$ with the following properties

$$\begin{cases} \gamma(t) = (\sin(t), \cos(t)) & \text{for } t \in [0, \frac{\pi}{2} - \epsilon], \\ \gamma(t) = (1 + \frac{\pi}{2} - \ell(\epsilon) - t, 0) & \text{for } t \in [\frac{\pi}{2} + \ell(\epsilon) - \epsilon, \frac{\pi}{2} + 1 + \ell(\epsilon) - 2\epsilon], \\ |\dot\gamma(t)| = 1 & \text{for all } t \in [0, \frac{\pi}{2}], 0 < \ell(\epsilon) \leq 2\epsilon. \end{cases}$$

where ℓ is the length of the rounded corner, which we require to be smaller than the distance (=2ϵ) the unrounded corner would cover. Now take the surface of revolution

$$S = \{(\gamma^1(t)\cos(s), \gamma^1(t)\sin(s), \gamma^2(t)) \,:\, s \in \mathbb{R}/2\pi\mathbb{R}, t \in I\}$$

using this γ. This surface agrees with the sphere for $t \in [0, \frac{\pi}{2} - \epsilon]$ and the disk for $t \in [\frac{\pi}{2} - \epsilon + \ell(\epsilon), \frac{\pi}{2} + 1 - 2\epsilon + \ell(\epsilon)]$ with the corners along the edge of the disc rounded.

From part a), we know that the set $U := \{(x, y, z) \in S : z \geq \cos(\frac{\pi}{2} - \epsilon)\}$ is geodesically convex. Now take the points $p = (\sin(\frac{\pi}{2} - \epsilon), 0, \cos(\frac{\pi}{2} - \epsilon))$ and $q = (-\sin(\frac{\pi}{2} - \epsilon), 0, \cos(\frac{\pi}{2} - \epsilon)$. Then we know from exercise 2 of sheet 8, that meridians are always geodesics. (Alternatively use exercise 5 of this sheet.) Thus the curve $t \mapsto (-\sin(t), 0, \cos(t))$ for $t \in [-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon]$ is the unique geodesic γ connecting p to q in U. This geodesic has length $\pi - 2\epsilon$. The other geodesic given by following $t \mapsto (\gamma^1(t), 0, \gamma^2(t))$ for $t \in [\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + 1 - 2\epsilon + \ell(\epsilon)]$ and then following $t \mapsto (-\gamma^1(2c - t), 0, \gamma^2(2c - t))$ for $t \in [c, c + 1 + \ell(\epsilon) - \epsilon]$ where $c := \frac{\pi}{2} + 1 - 2\epsilon + \ell(\epsilon)$. Thereby, we connect p to q via a geodesic of length $2 - 2\epsilon + \ell(\epsilon) \leq 2 < \pi - 2\epsilon$. So γ was not of minimal length.

c) Let $M = S^2$ and $U = M$. Then any two points p, q on M can be connected by a minimal geodesic, but such a geodesic is not necessarily unique. Take for example the north p_N and south pole p_S, then any meridian going through p_N and p_S is a geodesic that connects p_N and p_S and they also minimize distance.

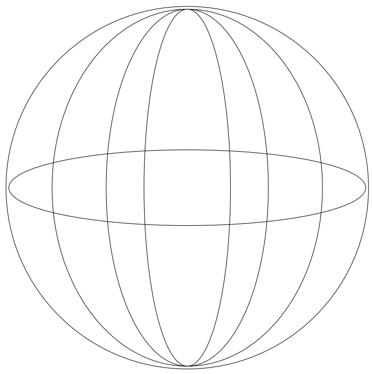

Figure 1: The meridians are geodesics connecting the south and the north pole

5. Let $M \subset \mathbb{R}^3$ be a two dimensional manifold and suppose that M is invariant under the (orthogonal) reflection across some plane $E \subset \mathbb{R}^3$. Show that E intersects M in a union of geodesics. Conclude for example that the coordinate planes intersect the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ in geodesics.

Hint: If $E \cap M$ would not be the image of a geodesic, then there would be points $p, q \in M$ very close to one another joined by two distinct minimal geodesics.

Solution: Suppose that $E \cap M$ is not the union of images of geodesics and denote by r_E the reflection across the plane E in $\mathbb{R}^3$.

Take $p \in M \cap E$ at a non-geodesic connected component of the intersection and let $U_r(p)$ be a geodesically convex neighbourhood of p as in exercise 3. Take $q \in U \cap E$ with $q \neq p$. This is possible, because otherwise p is an isolated point in $E \cap M$, which is a (constant) geodesic. Thus let γ be the unique geodesic connecting p and q in U. By assumption, image$(\gamma) \not\subset E$. So $r_E \circ \gamma$ is different from γ but has the same length as γ. Therefore,

$r_E \circ \gamma$ is a second geodesic which connects p and q in $r_E(U_r(p)) = U_r(p)$, which leads to a contradiction.

For the ellipsoid, the planes $E = \{x = 0\}$, $E = \{y = 0\}$ and $E = \{z = 0\}$ are all symmetry planes for the ellipsoid. So the three curves from the intersections of E with the ellipsoid are all images of geodesics.

6. Let $\gamma : I = [a, b] \to M$ be a smooth curve such that $\dot\gamma(t) \neq 0$ for all $t \in I$.

a) Define $\sigma : [a, b] \to [0, L(\gamma)]$ by

$$\sigma(t) := \int_a^t |\dot\gamma(s)|\, ds.$$

Prove that σ is a smooth diffeomorphism and that

$$\gamma' := \gamma \circ \sigma^{-1} : [0, L(\gamma)] \to M$$

is parametrized by arclength, i.e. $|\dot\gamma'(t)| = 1$ for all $t \in [0, T]$.

b) Prove that the derivative of the length functional at γ is given by

$$dL(\gamma)X = -\int_a^b \left\langle \dot V(t), X(t) \right\rangle dt, \qquad V(t) := \frac{\gamma(t)}{||\dot\gamma(t)||}$$

where $X \in \mathrm{Vect}(\gamma)$ with $X(a) = 0 = X(b)$.

c) Prove that γ is an extremal point of L if and only if the curve γ' from part a) is a geodesic.

Hint: Part a) provides a canonical parametrization for a geometric curve. Part b) depends on the assumption $\dot\gamma(t) \neq 0$ and the calculation is similar to the proof of Theorem 4.1.4. Part c) claims that a critical points of the length functional depend only on the geometrical curve and that the arclength parametrization yields a geodesic.

Solution:

a) The norm function $||\cdot|| : \mathbb{R}^k \to [0, \infty)$ is a smooth function on $\mathbb{R}^k \backslash \{0\}$. For a smooth curve $\gamma(t)$ with $\dot\gamma(t) \neq 0$, the composition $t \mapsto ||\dot\gamma(t)||$ is also smooth, and thus the integral function

$$\sigma : I \to \mathbb{R}, \qquad \sigma(t) := \int_a^t ||\dot\gamma(s)||\, ds$$

is smooth. Since $\dot\sigma(t) = ||\dot\gamma(t)|| > 0$, it follows that σ is strictly monotone increasing and a diffeomorphism onto its range $\sigma(I) = [0, L(\gamma)]$.

Define $\gamma' := \gamma \circ \sigma^{-1} : [0, L(\gamma)] \to M$. It follows from the chain rule (and the formula for the derivative of the inverse function) that

$$\dot\gamma'(t) = \dot\gamma(\sigma^{-1}(t)) \cdot \frac{1}{\dot\sigma(\sigma^{-1}(t))} = \dot\gamma(\sigma^{-1}(t)) \cdot \frac{1}{||\dot\gamma(\sigma^{-1}(t))||}$$

and this yields $||\dot\gamma'(t)|| = 1$ for all $t \in [0, L(\gamma)]$

b) Let $\gamma_s : [a,b] \to M$ be a smooth family of curves with $\gamma_0(t) = \gamma(t)$ and $\partial_s \gamma_s(t)|_{s=0} = X(t)$. Then

$$\begin{aligned}
dL(\gamma)X &= \frac{d}{ds}\bigg|_{s=0} L(\gamma_s) = \frac{d}{ds}\bigg|_{s=0} \int_a^b ||\dot\gamma_s(t)||\, dt \\
&= \int_a^b \frac{d}{ds}\bigg|_{s=0} \sqrt{\langle \dot\gamma_s(t), \dot\gamma_s(t)\rangle}\, dt \\
&= \int_a^b \frac{\langle \dot\gamma_s(t), \partial_s \dot\gamma_s(t)\rangle}{||\dot\gamma_s(t)||}\bigg|_{s=0} dt \\
&= \int_a^b \left\langle \frac{\dot\gamma(t)}{||\dot\gamma(t)||}, \dot X(t) \right\rangle dt \\
&= \int_a^b \left\langle V(t), \dot X(t) \right\rangle dt \\
&= \int_a^b \left\langle \dot V(t), X(t) \right\rangle dt
\end{aligned}$$

The last step follows from integration by parts and uses that the boundary terms vanish as $X(a) = 0 = X(b)$.

c) Let $\gamma' := \gamma \circ \sigma^{-1} : [0, L(\gamma)] \to M$ be defined as in a) and let $V \in \mathrm{Vect}(\gamma)$ and $V' \in \mathrm{Vect}(\gamma')$ be given by $V(t) := \dot\gamma(t)/||\dot\gamma(t)||$ and $V'(t) := \dot\gamma'(t)/||\dot\gamma'(t)||$. It follows from our calculation in part a) that

$$V'(t) = \dot\gamma'(t) = \frac{\dot\gamma(\sigma^{-1}(t))}{||\dot\gamma(\sigma^{-1}(t))||} = V(\sigma^{-1}(t)).$$

and thus

$$\dot V'(t) = \dot V(\sigma^{-1}(t)) \cdot \frac{1}{\dot\sigma(\sigma^{-1}(t))} = \dot V(\sigma^{-1}(t)) \frac{1}{||\dot\gamma(\sigma^{-1}(t))||}.$$

The formula in b) shows that γ is a critical point of L if and only if $\dot V(t) \in T_{\gamma(t)}M^\perp$ for all $t \in I$. This is equivalent to $\dot V'(t) \in T_{\gamma'(t)}M^\perp$ for all $t \in [0, L(\gamma)]$. By definition of the covariant derivative, this is equivalent to $\nabla V'(t) = 0$. Since $V'(t) = \dot\gamma'(t)$, this is indeed the geodesic equation for γ', and thus γ is a critical point if and only if γ' is a geodesic.