Solution 10

1. **a)** Let $M \subset \mathbb{R}^n$ be a m-dimensional manifold. Let $\psi : \Omega \to U \subset M$ be a local parametrisation. For $x \in \Omega$ define $b_i(x) = \frac{\partial \psi}{\partial x^i}(x)$ and $g_{ij}(x) := \langle b_i(x), b_j(x) \rangle$. Define $R^\ell_{ijk} : \Omega \to \mathbb{R}$ by

$$R(b_i, b_j)b_k =: \sum_{\ell=1}^m R^\ell_{ijk} b_\ell.$$

Show that

$$R^\ell_{ijk} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \sum_{\nu=1}^m (\Gamma^\ell_{i\nu}\Gamma^\nu_{jk} - \Gamma^\ell_{j\nu}\Gamma^\nu_{ik})$$

for all $\ell, i, j, k \in \{1, \cdots, m\}$ where Γ^k_{ij} are the usual Christoffel symbols, defined by $\nabla_{b_i} b_j = \sum_{k=1}^m \Gamma^k_{ij} b_k$.

b) Calculate R^ℓ_{ijk} for the stereographic projection on the sphere S^2.

Hint: For b), recall that we calculated in Exercise 4 of sheet 6, the metric in stereographic projection to be $g_{ij}(x) = \frac{4\delta_{ij}}{(1+|x|^2)^2}$. You can use the formulae for the Christoffel symbols from Exercise 6 of sheet 6. You only need to calculate R^1_{122} as the other components follow by symmetry.

Solution:

a) Let $b_i = \frac{\partial \psi}{\partial x_i} : \Omega \to \mathbb{R}^n$. We use the global formula for the curvature (5.2.12),

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Putting $X = b_i$, $Y = b_j$ and $Z = b_k$ we get

$$R(b_i, b_j)b_k = \nabla_{b_i} \nabla_{b_j} b_k - \nabla_{b_j} \nabla_{b_i} b_k,$$

as the comutator $[b_i, b_j] = \partial_i \partial_j \psi - \partial_j \partial_i \psi = 0$. Hence we have

$$\begin{aligned}
R(b_i, b_j)b_k &= \sum_{\ell=1}^m (\nabla_{b_i}(\Gamma^\ell_{jk} b_\ell) - \nabla_{b_j}(\Gamma^\ell_{ik} b_\ell)) \\
&= \sum_{\ell=1}^m (\partial_i \Gamma^\ell_{jk} b_\ell + \Gamma^\ell_{jk} \nabla_{b_i} b_\ell - \partial_j \Gamma^\ell_{ik} b_\ell - \Gamma^\ell_{ik} \nabla_{b_j} b_\ell) \\
&= \sum_{\ell=1}^m (\partial_i \Gamma^\ell_{jk} b_\ell - \partial_j \Gamma^\ell_{ik} b_\ell + \sum_{\nu=1}^m (\Gamma^\ell_{jk}\Gamma^\nu_{i\ell} - \Gamma^\ell_{ik}\Gamma^\nu_{j\ell}) b_\nu) \\
&= \sum_{\ell=1}^m \Big(\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \sum_{\nu=1}^m (\Gamma^\nu_{jk}\Gamma^\ell_{i\nu} - \Gamma^\nu_{ik}\Gamma^\ell_{j\nu}) \Big) b_\ell.
\end{aligned}$$

The second equality holds as the covariant derivative satisfies the Leibniz rule (5.2.7)

$$\nabla_X(fY) = f\nabla_X Y + (\mathcal{L}_X f)Y.$$

b) Recall that we have from exercise 6 of sheet 6,

$$\begin{aligned}
\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2\lambda}\frac{\partial \lambda}{\partial x^1} = -\frac{2x_1}{1+|x|^2}, \\
\Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2\lambda}\frac{\partial \lambda}{\partial x^2} = -\frac{2x_2}{1+|x|^2},
\end{aligned}$$

where $\lambda(x) = \frac{4}{(1+|x|^2)^2}$. Thus

$$\begin{aligned} R_{122}^1 &= \partial_1 \Gamma_{22}^1 - \partial_2 \Gamma_{12}^1 + \Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{12}^1 \Gamma_{21}^1 + \Gamma_{22}^2 \Gamma_{12}^1 - \Gamma_{12}^2 \Gamma_{22}^1 \\ &= \partial_1 \frac{2x_1}{1+|x|^2} + \partial_2 \frac{2x_2}{1+|x|^2} + 0 = \frac{4}{(1+|x|^2)^2} = \lambda. \end{aligned}$$

We now use the symmetry relations for the Riemann tensor and $R_{ijk\ell} = \lambda R_{ijk}^\ell$ to conclude that

$$R_{211}^2 = -R_{121}^2 = -R_{212}^1 = R_{122}^1 = \lambda$$

and that all the other twelve symbols are zero.

As a reality check, we have that $K(\psi(x)) = \frac{\langle R(b_1,b_2)b_2,b_1\rangle}{|b_1|^2|b_2|^2} = \frac{\lambda^2}{\lambda^2} = 1$ which means that the round sphere has constant Gaussian curvature equal to 1.

2. Let $G \subset O(n)$ be a Lie subgroup of $O(n)$.

a) Every $\xi \in T_{\mathbb{1}}G = \mathrm{Lie}(G)$ determines a left-invariant vector field on G by the formula $X_\xi(g) := g\xi \in T_gG$. Show that

$$\nabla_{X_\xi} X_\eta = \frac{1}{2} X_{[\xi,\eta]}$$

where $[\xi, \eta] = \xi\eta - \eta\xi$ is the commutator Lie bracket.

b) Show that $[X_\xi, X_\eta] = X_{[\eta,\xi]}$.

c) Show that the Riemann curvature tensor of G is given by

$$R_g(g\xi, g\eta)g\zeta = -\frac{1}{4} g[[\xi,\eta],\zeta]$$

for $g \in G$ and $\xi, \eta, \zeta \in T_{\mathbb{1}}G$.

Hint: For a): Prove the formula first for $G = O(n)$ using Exercise 6 from Exercise Sheet 8. Why does it remain valid for any subgroup? For b): Use that ∇ is torsion free. For c): Extend the tangent vectors to left-invariant vector fields and use the relations from a) and b) to calculate the curvature tensor.

Solution:

a) Assume first $G = O(n)$. We have seen in Exercise 6 from Exercise Sheet 8 that the orthogonal projection from $\mathbb{R}^{n\times n}$ onto $T_gO(n)$ is given by

$$\Pi(g)\beta := \frac{1}{2}(\beta - g\beta^T g).$$

Since $dX_\eta(g)\hat{g} = \hat{g}\eta$, it follows

$$\nabla_{\hat{g}} X_\eta(g) = \Pi(g) dX_\eta(g)\hat{g}\eta = \Pi(g)\hat{g}\eta = \frac{1}{2}\left(\hat{g}\eta - g\eta^\top \hat{g}^\top g\right)$$

and with $\hat{g} = X_\xi(g) = g\xi$ this yields

$$\nabla_{X_\xi} X_\eta(g) = \frac{1}{2}\left(g\xi\eta - g\eta^\top\xi^\top g^\top g\right) = \frac{1}{2} g(\xi\eta - \eta\xi) = \frac{1}{2} X_{[\xi,\eta]}(g).$$

Here we used that $g^\top g = \mathbb{1}$ for $g \in O(n)$ and $\xi = -\xi^\top$ and $\eta = -\eta^\top$ for $\xi, \eta \in T_{\mathbb{1}}O(n)$.

We claim that the formula for the covariant derivative remains valid for any Lie subgroup $G \subset O(n)$: Let $\xi, \eta \in T_{\mathbb{1}}G$. First recall that $[\xi, \eta] \in T_{\mathbb{1}}G$, since G is a

Lie group, and thus $X_{[\xi,\eta]}(g) \in T_gG$ for all $g \in G$. Denote by $\Pi^G(g)$ the orthogonal projection onto T_gG and by $\Pi^{O(n)}(g)$ the orthogonal projection onto $T_gO(n)$. They are compatible in the sense that $\Pi^G(g) = \Pi^G(g) \circ \Pi^{O(n)}(g)$. It now follows

$$\begin{aligned}\nabla_{X_\xi} X_\eta(g) &:= \Pi^G(g) dX_\eta(g) X_\xi(g) \\ &= \Pi^G(g) \circ \Pi^{O(n)}(g) dX_\eta(g) X_\xi(g) \\ &= \Pi^G(g) \frac{1}{2} X_{[\xi,\eta]}(g) \\ &= \frac{1}{2} X_{[\xi,\eta]}(g).\end{aligned}$$

The penultimate equation follows from our calculation for $O(n)$ and the last equation uses $X_{[\xi,\eta]}(g) \in T_gG$.

b) Since the Levi-Civita connection is torsion free, it satisfies $[X, Y] = \nabla_Y X - \nabla_X Y$. In particular

$$[X_\xi, X_\eta] = \nabla_{X_\eta} X_\xi - \nabla_{X_\xi} X_\eta = \frac{1}{2} X_{[\eta,\xi]} - \frac{1}{2} X_{[\xi,\eta]} = X_{[\xi,\eta]}.$$

c) We can calculate the curvature directly using the relations from part a):

$$\begin{aligned}R_g(g\xi, g\eta)g\zeta &:= \nabla_{X_\xi} \nabla_{X_\eta} X_\zeta(g) - \nabla_{X_\eta} \nabla_{X_\xi} X_\zeta(g) + \nabla_{[X_\xi,X_\eta]} X_\zeta(g) \\ &= \frac{1}{2} \nabla_{X_\xi} X_{[\eta,\zeta]}(g) - \frac{1}{2} \nabla_{X_\eta} X_{[\xi,\zeta]}(g) - \nabla_{X_{[\xi,\eta]}} X_\zeta(g) \\ &= \frac{1}{4} X_{[\xi,[\eta,\zeta]]}(g) - \frac{1}{4} X_{[\eta,[\xi,\zeta]]}(g) - \frac{1}{2} X_{[[\xi,\eta],\zeta]}(g) \\ &= \frac{1}{4} g\left([\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]]\right) - \frac{1}{2} g[[\xi,\eta],\zeta] \\ &= -\frac{1}{4} g[\zeta,[\xi,\eta]] - \frac{1}{2} g[[\xi,\eta],\zeta] \\ &= -\frac{1}{4} g[[\xi,\eta],\zeta].\end{aligned}$$

The penultimate equations uses the Jacobi identity $[\xi,[\eta,\zeta]]+[\eta,[\zeta,\xi]]+[\zeta,[\xi,\eta]] = 0$.

3. Let $X \in \mathrm{Vect}(M)$. Prove the following.

a) If the map $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ satisfies

$$\mathcal{L}_X \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle,$$

for all $Y, Z \in \mathrm{Vect}(M)$, then D_X is linear.

b) If the map $D : \mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto D_X$ satisfies

$$D_Y X - D_X Y = [X, Y],$$

for all $X, Y \in \mathrm{Vect}(M)$, then D is linear.

Solution:

a) Suppose that $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ satisfies the following

$$\mathcal{L}_X \langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle, \tag{1}$$

for all vector fields Y, Z. We need to prove that $D_X(\alpha Y_1 + \beta Y_2) = \alpha D_X Y_1 + \beta D_X Y_2$. Substituting in the equation (1) we get

$$\mathcal{L}_X \langle \alpha Y_1 + \beta Y_2, Z\rangle = \langle D_X(\alpha Y_1 + \beta Y_2), Z\rangle + \langle \alpha Y_1 + \beta Y_2, D_X Z\rangle.$$

As both Lie derivative and scalar product are linear we get

$$\begin{aligned}\langle D_X(\alpha Y_1 + \beta Y_2), Z\rangle &= \alpha\left(\mathcal{L}_X\langle Y_1, Z\rangle - \langle Y_1, D_X Z\rangle\right) + \beta\left(\mathcal{L}_X\langle Y_2, Z\rangle - \langle Y_2, D_X Z\rangle\right)\\ &= \alpha\langle D_X Y_1, Z\rangle + \beta\langle D_X Y_2, Z\rangle\\ &= \langle \alpha D_X Y_1 + \beta D_X Y_2, Z\rangle.\end{aligned}$$

As the previous equality holds for all $Z \in Vect(M)$ we conclude

$$D_X(\alpha Y_1 + \beta Y_2) = \alpha D_X Y_1 + \beta D_X Y_2.$$

b) Suppose that the map $\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto D_X$ satisfies

$$D_Y X - D_X Y = [X, Y], \tag{2}$$

for all vector fields X and Y on M. Let $Y = \alpha Y_1 + \beta Y_2$. Substituting in the equality (2) we get

$$D_{\alpha Y_1 + \beta Y_2} X - D_X(\alpha Y_1 + \beta Y_2) = [X, \alpha Y_1 + \beta Y_2].$$

We have that D_X is linear i.e. $D_X(\alpha Y_1 + \beta Y_2) = \alpha D_X Y_1 + \beta D_X Y_2$ and the Lie bracket of vector fields is linear as well. Hence it follows that

$$\begin{aligned}D_{\alpha Y_1 + \beta Y_2} X &= \alpha(D_X Y_1 + [X, Y_1]) + \beta(D_X Y_2 + [X, Y_2])\\ &= \alpha D_{Y_1} X + \beta D_{Y_2} X.\end{aligned}$$

4. **a)** Prove that every isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map

$$\phi(p) = \Phi p + p_0,$$

where $p_0 \in \mathbb{R}^n$ and $\Phi \in O(n)$. Thus ϕ is a composition of rotation and translation.

b) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Suppose $M,\ M' \subset \mathbb{R}^n$ are manifolds such that $\phi(M) = M'$. Show that ϕ intertwines their second fundamental form:

$$(d\phi(p))^{-1} h'_{\phi(p)}(d\phi(p)v, d\phi(p)w) = h_p(v, w), \quad \forall v, w \in T_pM, \quad \forall p \in M.$$

Hint: For b): Assume that $\phi(p) = \Phi p + p_0$ and show first that the families of orthogonal projections on M and M' are related by $\Pi'(\phi(p)) = \Phi^t \Pi(p) \Phi$.

Solution:

a) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and define

$$p_0 := \phi(0), \qquad v_i := \phi(e_i) - p_0 \quad \text{for } i = 1, \dots, n.$$

We claim that $v_1, \dots, v_n$ is an orthonormal basis of $\mathbb{R}^n$. Since ϕ is an isometry, we have

$$||v_i|| = ||\phi(e_i) - \phi(0)|| = ||e_i - 0|| = 1$$

and

$$||v_i - v_j||^2 = ||\phi(e_i) - \phi(e_j)||^2 = ||e_i - e_j||^2 = 2(1 - \delta_{ij}).$$

From these two equation we conclude

$$\langle v_i, v_j \rangle = \frac{1}{2}\left(||v_i||^2 + ||v_2||^2 - ||v_i - v_j||^2\right) = \delta ij$$

and hence $v_1, \dots, v_n$ is an orthonormal basis.

For $p \in \mathbb{R}^n$, this yields the representation

$$\phi(p) = p_0 + \phi(p) - p_0 = p_0 + \sum_{i=1}^{n} \langle \phi(p) - p_0, v_i \rangle v_i$$

and similar to the calculation above, it holds

$$\begin{aligned}
\langle \phi(p) - p_0, v_i \rangle &= \frac{1}{2}\left(||\phi(p)||^2 + ||\phi(e_i) - \phi(0)||^2 - ||\phi(p) - \phi(e_i)||^2\right) \\
&= \frac{1}{2}\left(||p||^2 + 1 - ||p - e_i||^2\right) \\
&= \frac{1}{2}\left(p_i^2 + 1 - (p_i - 1)^2\right) \\
&= p_i
\end{aligned}$$

Hence $\phi(p) = p_0 + \sum_{i=1}^{n} p_i v_i = p_0 + \Phi p$ and this proves the claim.

b) By part a), we have $\phi(p) = p_0 + \Phi p$ for some $p_0 \in \mathbb{R}^n$ and $\Phi \in \mathrm{O}(n)$. As $\phi|_M : M \to M'$ is an isometry, it follows that

$$d\phi(p) : T_pM \to T_{\phi(p)}M', \qquad v \mapsto \Phi v$$

is an isomorphism. It follows from Exercise 5 on Exercise Sheet 4 that the families of orthogonal projections on M and M' are related by

$$\Pi'(\phi(p)) = \Phi \Pi(p) \Phi^t.$$

Indeed, one checks readily that $\Pi'(\phi(p)) = \Pi'(\phi(p))^2 = \Pi'(\phi(p))^\top$ and $\mathrm{Im}(\Pi'(\phi(p))) = \Phi T_pM = T_{\phi(p)}M'$ and these properties characterize $\Pi'(\phi(p))$ uniquely.

Using $\Phi \in \mathrm{O}(m)$, the formula is equivalent to $\Pi(p) = \Phi^\top \Pi'(\phi(p))\Phi$. Differentiating this identity at $p \in M$ in direction $v \in T_pM$ yields

$$d\Pi(p)v = \Phi^\top \left(d\Pi'(\phi(p))\Phi v\right) \Phi.$$

Evaluation of both sides on $w \in T_pM$ yields

$$h_p(v, w) = (d\Phi(p)v)\, w = \Phi^\top \left(d\Pi'(\phi(p))\Phi v\right) \Phi w = \Phi^\top h'_{\phi(p)}(\Phi v, \Phi w).$$

This is the desired formula as $\Phi = d\phi(p)$ and $\Phi^\top = \Phi^{-1}$.

5. A complete vector field $X \in \mathrm{Vect}(M)$ is called a **Killing vector field** if its flow $\phi^t : M \to M$ is an isometry for every $t \in \mathbb{R}$.

a) Give examples of Killing vector fields.

b) Let X be a Killing vector field. Prove that

$$\langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0,$$

for $v, w \in T_pM,\ p \in M$.

c) Let X be a Killing field and $\gamma : I \to M$ a geodesic. Show that

$$\frac{d}{dt}\langle \dot\gamma, X(\gamma)\rangle = 0.$$

Hint: For a), you can try to find Killing fields for $M = \mathbb{R}^n$.

Solution:

a) For $M = \mathbb{R}^n$, take $X(p) = A \cdot p$, where $A \in \mathbb{R}^{n\times n}$ satsifies $A^\top = -A$. Then follows $A \in T_{\mathbb{1}}O(n)$ and it follows from Exercise 6 on Exercise Sheet 8 that $\exp(tA) \in O(n)$. From this follows that the flow of X given by

$$\phi^t(p) = e^{tA}p$$

is an isomoetry for every $t \in \mathbb{R}^n$. Note that these vector fields are tangential to S^n and restrict to yields Killing vector fields on S^n.

The constant vector field $X(p) = p_0$ on $M = \mathbb{R}^n$ is a Killing field generating translations.

b) The identity

$$\langle v, w\rangle = \langle d\phi^t(p)v, d\phi^t(p)w\rangle$$

is valid for all $t \in \mathbb{R}$, $p \in M$ and $v, w \in T_pM$. Differentiating this identity yields

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \langle d\phi^t(p)v, d\phi^t(p)w\rangle \\
&= \left\langle \left.\frac{d}{dt}\right|_{t=0} d\phi^t(p)v, d\phi^0(p)w \right\rangle + \left\langle d\phi^0(p)v, \left.\frac{d}{dt}\right|_{t=0} d\phi^t(p)w \right\rangle \\
&= \langle dX(p)v, w\rangle + \langle v, dX(p)w\rangle = \\
&= \langle \Pi(p)(dX(p)v), w\rangle + \langle v, \Pi(p)(dX(p)w)\rangle \\
&= \langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle
\end{aligned}$$

where the penultimate equation uses $u, v \in T_pM$.

c) We have

$$\frac{d}{dt}\langle \dot\gamma, X(\gamma(t))\rangle = \langle \nabla\dot\gamma, X(\gamma)\rangle + \langle \dot\gamma, \nabla_{\dot\gamma} X\rangle.$$

Now $\langle \nabla\dot\gamma, X(\gamma)\rangle = 0$ as γ is a geodesic satisfying the equation $\nabla\dot\gamma = 0$. It follows from part b) with $v = w = \dot\gamma$ that $\langle \dot\gamma, \nabla_{\dot\gamma} X\rangle = 0$ and this proves the claim.

6. Prove the following.

a) Every compact connected $1-$ manifold is diffeomorphic to S^1.

b) Define the length of a compact connected $1-$manifold.

c) Two compact connected 1−manifolds are isometric if and only if they have the same length.

Hint: For a): Let $\gamma : \mathbb{R} \to M$ be a geodesic with $|\dot\gamma| = 1$. First, show that γ is not injective. Second, show by contradiction for $t_0 < t_1$ with $\gamma(t_0) = \gamma(t_1)$ that $\dot\gamma(t_0) = \dot\gamma(t_1)$ (otherwise prove that $\gamma(t_0 + t) = \gamma(t_1 - t)$ for all $t \in \mathbb{R}$).

Solution:

a) Solution I:

Step 1: Suppose for a contradiction that the geodesic $\gamma : \mathbb{R} \to M$ with $|\dot\gamma| = 1$ is injective. As $\dot\gamma(t) \neq 0$, we have that γ is a parametrisation and coincides with the exponential map $\exp_{\gamma(0)}$. Thus the injectivity radius $\mathrm{inj}(\gamma(0), M)$ of M would be infinite. As M is a compact manifold, the diameter

$$\ell := \mathrm{diam}(M) := \sup_{x,y\in M} d(x, y)$$

with respect to the intrinsic distance is finite. However, as the exponential map minimises distance up to its injectivity radius,

$$d(\gamma(0), \gamma(\ell + 1)) = \ell + 1,$$

which is a contradiction.

Step 2: Now take $t_0 < t_1$ such that $\gamma(t_0) = \gamma(t_1)$. We shall prove that in this case

$$\dot\gamma(t_0) = \dot\gamma(t_1).$$

Suppose that $\dot\gamma(t_0) = -\dot\gamma(t_1)$. Then for all $t \in [0, t_1 - t_0]$ the geodesic γ satisfies

$$\gamma(t_0 + t) = \gamma(t_1 - t).$$

For small t, this follows from the uniqueness of the geodesic. But the set $t \in [0, t_1 - t_0]$ such that $\gamma(t_0 + t) = \gamma(t_1 - t)$ is open and closed in $[0, t_1 - t_0]$, hence it will hold on the whole $[0, t_1 - t_0]$. Now we have a contradiction as $\gamma(t_0 + t) = \gamma(t_1 - t)$ around $t = \frac{t_1-t_0}{2}$ implies that $\dot\gamma(\frac{t_0+t_1}{2}) = 0$.
Let T be the smallest positive number such that $\gamma(t_0) = \gamma(t_0 + T)$. Then

$$\phi : S^1 \to M, \qquad \phi(e^{2\pi \mathbf{i}\theta}) = \gamma(\theta T)$$

is the required diffeomorphism.

Solution II:

As M is compact we can cover it with finitely many charts / parametrisations

$$\phi_i : I_i \to U_i \subset M, \qquad i = 1, \cdots , n$$

where $I_i \subset \mathbb{R}$ are open intervals. One can also suppose that all these parametrisations are given by arc-length i.e. the vector $d\phi_i(x)1$ is unit vector in $T_{\phi_i(x)}M$. In the case that $U_i \cap U_j \neq \emptyset$ there are two possibilities:

- $U_i \cap U_j$ is connected, i.e. it has only one connected component. In this case we can merge these two parametrization into one parametrization, i.e. ϕ_i extends to a parametrization of $\phi_i(I_i) \cup \phi_j(I_j)$.

- In the case $U_i \cap U_j$ has two components then $M \subset U_i \cup U_j$ and M must be diffeomorphic to S^1.

As M is compact and connected it must happen that in some step we end up in the second case - $U_i \cap U_j$ has two connected components (otherwise M would be open). For the proof of these facts we refer to J. Milnor's book - Topology from differential view point.

b) In the previous part we have seen that there exists a diffeomorphism $\phi : S^1 \to M$. The length of M is defined as

$$L(M) = \int_{S^1} |\dot{\phi}(\theta)| d\theta = \int_0^1 |\dot{\phi}(e^{2\pi \mathbf{i}\theta})|\, d\theta.$$

The previous expression does not depend on the diffeomorphism ϕ. Indeed, take any other $\phi' = \phi(\alpha(t))$ and use the change of variables to prove that the integral is the same.

Suppose $\gamma : \mathbb{R} \to M$ is a geodesic with $|\dot{\gamma}(t)| = 1$. Using the diffeomorphism constructed from γ in a) we find that $L(M) = T$ where T is the smallest positive number such that $\gamma(t_0) = \gamma(t_0 + T)$.

c) If $f : M \to N$ is an isometry and $\phi : S^1 \to M$ is a diffeomorphism then

$$\gamma' : S^1 \to N,\ \phi'(t) = f(\phi(t))$$

is a diffeomorphism and

$$L(N) = \int_0^1 |\dot{\phi}'(t)| dt = \int_0^1 |df(\phi(t))\dot{\phi}(t)| dt = \int_0^1 |\dot{\phi}(t)| dt = L(M).$$

For the converse, let $\gamma : \mathbb{R} \to M$ and $\gamma' : \mathbb{R} \to N$ be geodesics with $|\dot{\gamma}(t)| = 1 = |\dot{\gamma}'(t)|$. It follows from part a) and b) that for $T := L(N) = L(M)$ the maps

$$\phi : S^1 \to M, \qquad \phi(e^{2\pi \mathbf{i}\theta}) = \gamma(\theta T)$$

$$\phi' : S^1 \to M', \qquad \phi(e^{2\pi \mathbf{i}\theta}) = \gamma'(\theta T)$$

are well-defined diffeomorphisms. The composition $\phi' \circ \phi^{-1} : M \to N$ is clearly a diffeomorphism and the relation

$$d\phi'(t) \circ d\phi(t)^{-1} \dot{\gamma}(t) = \dot{\gamma}'(t)$$

implies $|d(\phi' \circ \phi^{-1})| = 1$ and hence $\phi' \circ \phi^{-1}$ is an isometry.