Solution 11

1. Let $\Sigma \subset \mathbb{R}^3$ be a 2-manifold and $p_0 \in \Sigma$. Let $0 < \epsilon < \mathrm{inj}(p_0, \Sigma)$. Let $\gamma_\epsilon : [0,1] \to \Sigma$ be a parametrisation of the geodesic circle $C_\epsilon := \{p \in \Sigma : d(p, p_0) = \epsilon\}$ of radius $\epsilon > 0$ around p_0. Define $\ell_\epsilon := \mathrm{L}(\gamma_\epsilon)$ to be the length of γ_ϵ.

a) Prove that the following relation holds.

$$K(p_0) = -3 \left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \frac{\ell_\epsilon}{2\pi\epsilon}. \tag{1}$$

b) Use (1) for $\Sigma = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ and $\Sigma = S^2 \subset \mathbb{R}^3$ to calculate their Gauss curvature.

Hint: For a), this is a long calculation in geodesic normal coordinates around p_0. In these coordinates, we have that $g_{ij}(0) = \delta_{ij}$, $\partial_k g_{ij}(0) = 0$ and $\partial_{11} g_{22}(0) = \partial_{22} g_{11}(0) = -2\partial_{12} g_{12}(0)$. All other second derivatives $\partial_{ij} g_{k\ell}(0)$ are zero. Also we can express $R^1_{122}(0)$ in terms of derivatives of $\partial_{11} g_{22}(0)$. In geodesic normal coordinates C_ϵ can be easily parametrised. After that, calculate until your fingers burn.

Solution:

a) Let $0 < \epsilon_0 < \mathrm{inj}(M, p_0)$. We will work in geodesic normal coordinates on $B_{\epsilon_0}(0)$. Denote by g_{ij} the metric and recall that $g_{ij}(0) = \delta_{ij}$ and $\partial_k g_{ij}(0) = 0$. (which is equivalent to the Christoffel symbols $\Gamma^k_{ij}(0) = 0$.) Also as the curves $t \mapsto tv$ for $v \in B_{\epsilon_0}(0)$ is a geodesic, we have the relation $\sum_{i,j=1}^2 v^i v^j \Gamma^k_{ij}(tv) = 0$ for $k = 1, 2$ and all $t < 1$. Differentiating this equation and setting $t = 0$, we get $\sum_{i,j,\ell=1}^2 v^i v^j v^\ell \partial_\ell \Gamma^k_{ij}(0) = 0$. This expression is a polynomial in v^1 and v^2. For it to be zero, we need all the coefficients to be zero. This gives us the relations

$$\begin{aligned}
&\partial_1\Gamma^1_{11}(0) = \partial_1\Gamma^2_{11}(0) = \partial_2\Gamma^1_{22}(0) = \partial_2\Gamma^2_{22}(0) = 0,\\
&\partial_1\Gamma^1_{22}(0) + 2\partial_2\Gamma^1_{12}(0) = \partial_2\Gamma^1_{11}(0) + 2\partial_1\Gamma^1_{12}(0) = 0,\\
&\partial_1\Gamma^2_{22}(0) + 2\partial_2\Gamma^2_{12}(0) = \partial_2\Gamma^2_{11}(0) + 2\partial_1\Gamma^2_{12}(0) = 0.
\end{aligned}$$

Now using the relations $\Gamma^k_{ij} = \frac{1}{2}\sum_{\ell=1}^2 g^{k\ell}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})$, $\partial_k g^{ij}(0) = 0$ and $g^{ij} = \delta^{ij}$, we get the corresponding relations

$$\begin{aligned}
&\partial_{11}g_{11}(0) = \partial_{11}g_{12}(0) - \frac{1}{2}\partial_{12}g_{22}(0) = \partial_{22}g_{12}(0) - \frac{1}{2}\partial_{12}g_{11}(0) = \partial_{22}g_{22}(0) = 0\\
&\partial_{12}g_{12}(0) - \frac{1}{2}\partial_{11}g_{22}(0) + \partial_{22}g_{11}(0) = \frac{3}{2}\partial_{12}g_{11}(0) = 0\\
&\frac{3}{2}\partial_{12}g_{22}(0) = \partial_{12}g_{12}(0) - \frac{1}{2}\partial_{22}g_{11}(0) + \partial_{11}g_{22}(0) = 0
\end{aligned}$$

From these, we extract the following relations

$$\begin{aligned}
&\partial_{11}g_{11}(0) = \partial_{11}g_{12}(0) = \partial_{22}g_{22}(0) = \partial_{22}g_{12}(0) = \partial_{12}g_{11}(0) = \partial_{12}g_{22}(0) = 0,\\
&-2\partial_{12}g_{12}(0) = \partial_{11}g_{22}(0) = \partial_{22}g_{11}(0).
\end{aligned} \tag{2}$$

We also see from the formula of R in terms of Christoffel symbols from last week's exercise 1, that

$$\begin{aligned}
K(p_0) = R^1_{122}(0) &= -\partial_1\Gamma^2_{12} + \partial_2\Gamma^2_{11}\\
&= -\frac{1}{2}(\partial_{11}g_{22}(0) + \partial_{22}g_{11}(0) - 2\partial_{12}g_{12}(0)) = -\frac{3}{2}\partial_{11}g_{22}(0).
\end{aligned}$$

Now look at $\gamma_\epsilon(t) := \epsilon(\cos(t), \sin(t)) = \epsilon\gamma(t)$ for $t \in [0, 2\pi)$ and $\epsilon < \epsilon_0 < \mathrm{inj}(M, p_0)$. By a theorem on the injectivity radius, we know that $\mathrm{im}(\exp_{p_0} \circ \gamma_\epsilon) = C_\epsilon$. Therefore,

$\ell_\epsilon = \mathrm{L}(\gamma_\epsilon)$. Thus we calculate $\frac{\ell_\epsilon}{2\pi\epsilon} = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\sum_{i,j=1}^2 \dot\gamma^i g_{ij}(\epsilon\gamma)\dot\gamma^j}\,dt$. Differentiating yields

$$\frac{d}{d\epsilon}\frac{\ell_\epsilon}{2\pi\epsilon} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\sqrt{\sum_{i,j=1}^2 \dot\gamma^i g_{ij}(\epsilon\gamma)\dot\gamma^j}}\sum_{i,j,k=1}^2 \partial_k g_{ij}(\epsilon\gamma)\dot\gamma^i\dot\gamma^j\gamma^k\,dt,$$

$$\frac{d^2}{d\epsilon^2}\frac{\ell_\epsilon}{2\pi\epsilon} = \frac{1}{2\pi}\int_0^{2\pi}\frac{-1}{4\sqrt{\sum_{i,j=1}^2 \dot\gamma^i g_{ij}(\epsilon\gamma)\dot\gamma^j}^3}\left(\sum_{i,j=1}^2 \partial_k g_{ij}(\epsilon\gamma)\dot\gamma^i\dot\gamma^j\gamma^k\right)^2 dt$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\sqrt{\sum_{i,j=1}^2 \dot\gamma^i g_{ij}(\epsilon\gamma)\dot\gamma^j}}\sum_{i,j,k,\ell=1}^2 \partial_{k\ell} g_{ij}(\epsilon\gamma)\dot\gamma^i\dot\gamma^j\gamma^k\gamma^\ell\,dt,$$

For $\epsilon = 0$, we get

$$\lim_{\epsilon\to 0^+}\frac{\ell_\epsilon}{2\pi\epsilon} = 1, \qquad \frac{d}{d\epsilon}\bigg|_{\epsilon=0}\frac{\ell_\epsilon}{2\pi\epsilon} = 0,$$
$$-3\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0}\frac{\ell_\epsilon}{2\pi\epsilon} = -\frac{3}{4\pi}\sum_{i,j,k,\ell=1}^2\int_0^{2\pi}\partial_{k\ell}g_{ij}(0)\dot\gamma^i\dot\gamma^j\gamma^k\gamma^\ell\,dt$$
$$= -\frac{3}{4\pi}\left(\int_0^{2\pi}\sin^4(t) + 2\cos^2(t)\sin^2(t) + \cos^4(t)\,dt\right)\partial_{11}g_{22}(0)$$
$$= -\frac{3}{4\pi}2\pi\partial_{11}g_{22}(0) = K(p_0),$$

where the penultimate line is obtained by using the relation (2) together with the combinatorics $(\{11\},\{22\})$ and $(\{22\},\{11\})$ appear once each whereas $(\{12\},\{12\})$ appears four times. The last ingredient is that $\dot\gamma^i(t)\dot\gamma^j(t)\gamma^k(t)\gamma^\ell(t) = \sin^4(t)$, $-\cos(t)^2\sin(t)^2$, resp. $\cos^4(t)$ for indices $(11,22)$, $(12,12)$, resp. $(22,11)$. In the process, we proved the formula for circumference and area of a geodesic disk

$$\ell_\epsilon = 2\pi\epsilon - \pi\frac{K(p_0)}{3}\epsilon^3 + O(\epsilon^4),$$
$$A_\epsilon = \int_0^\epsilon \ell_\epsilon\,d\epsilon = \pi\epsilon^2 - \pi\frac{K(p_0)}{12}\epsilon^4 + O(\epsilon^5).$$

Intuitively, think of a pizza sliced in 6 pieces. If you take out one piece and glue it back together, you get less pizza and positive curvature at the centre. Whereas if you magically make a 7th piece appear and glue it in, then the pizza become slightly wiggly (negatively curved like a saddle) and obviously we have more pizza. Look at `http://www.theiff.org/oexhibits/oe1e.html` to see what it looks like if you put a 7th slice at each point. (Hyperbolic planes, i.e. local models for negatively curved surfaces, look rather odd in $\mathbb{R}^3$.)

Note that we were free to choose any coordinate chart for the calculation, but the reader may imagine that the calculations would become even longer and tedious for a general chart. This should indicate that normal geodesic coordinates simplify local calculations quite a lot in certain situations.

b) For $\mathbb{R}^2 \subset \mathbb{R}^3$, we have that $\ell_\epsilon = 2\pi\epsilon$ as we are in the standard Euclidean length. Thus $K(p) = 0$ for all $p \in \mathbb{R}^2$.

For the sphere, take an orthonormal basis $e_1, e_2 \in T_{p_0}S^2$. Then, we get C_ϵ is parametrised by $\gamma(t) = p_0\cos(\epsilon) + (\cos(t)e_1 + \sin(t)e_2)\sin(\epsilon)$. Thus $\ell_\epsilon = 2\pi\sin(\epsilon)$. Therefore,

$$\begin{aligned} K(p_0) &= -3\,\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0}\frac{\ell_\epsilon}{2\pi\epsilon} = -3\,\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0}\frac{\sin(\epsilon)}{\epsilon} \\ &= 3\lim_{\epsilon\to 0^+}\left(\frac{2(\epsilon\cos(\epsilon)-\sin(\epsilon))}{\epsilon^3} + \frac{\sin(\epsilon)}{\epsilon}\right) = 1. \end{aligned}$$

(Of course, Taylor expansion also works. What would happen to K if $\ell_\epsilon = 2\pi\sinh(\epsilon)$?)

2. Let ψ be a parametrisation of the (embedded) surface of revolution S, as in exercise 3 of sheet number 7.

a) Show that the Gauss curvature of S is given by

$$K(\psi(s,t)) = \frac{R^1_{122}(s,t)}{g_{22}(s,t)} = \frac{-\dot\gamma_2(t)^2\ddot\gamma_1(t) + \dot\gamma_1(t)\dot\gamma_2(t)\ddot\gamma_2(t)}{\gamma_1(t)(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)^2}$$

b) Use part a) to calculate the Gauss curvature of

(i) The cylinder $Z = S^1 \times \mathbb{R} \subset \mathbb{R}^3$

(ii) The sphere $S^2 \subset \mathbb{R}^3$.

(iii) The torus $T^2 = \{(x,y,z) \in \mathbb{R}^3 : \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 = a^2\}$ for $0 < a < R$.

c) The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (1/\cosh(t), t - \tanh(t))$, is called *tractrix*. Its associated surface of revolution is *Beltrami's pseudosphere*:

$$\Sigma := \left\{ \left(\frac{\cos(s)}{\cosh(t)}, \frac{\sin(s)}{\cosh(t)}, t - \tanh(t)\right) \,\middle|\, t \in \mathbb{R},\ s \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Use part a) to show that $K(\psi(s,t)) = -1$ provided $t \neq 0$.

Hint: For part a) Use Theorem 5.3.7 on the Gaussian curvature and exercise 1 of exercise sheet 10 on the curvature tensor in local coordinates. We have calculated the Christoffel symbols already in exercise 3 of exercise sheet 7.

Solution:

a) It follows from Theorem 5.3.7 on the Gaussian curvature that

$$K(p) = \frac{\langle R(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2} \qquad \forall u, v \in T_pM.$$

In local coordinates, using the notation from exercise 1 of exercise sheet 10, this yields

$$K(\psi(s,t)) = \frac{\sum_{m=1}^2 R^m_{122}(s,t)g_{m1}(s,t)}{g_{11}(s,t)g_{22}(s,t) - g_{12}(s,t)^2}.$$

We have seen in Exercise 3 on Exercise Sheet 7 that $g_{12}(s,t) = g_{21}(s,t) = 0$ and thus

$$K(\psi(s,t)) = \frac{R^1_{122}(s,t)}{g_{22}(s,t)}.$$

It follows from exercise 1 of exercise sheet 10 that

$$R^1_{122} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \sum_{\nu=1}^n \Gamma^1_{1\nu}\Gamma^n_{22} - \Gamma^1_{2\nu}\Gamma^\nu_{12}.$$

Using the expressions for the Christoffel symbols obtained in exercise 3 of exercise Sheet 7, this yields

$$\begin{aligned} R^1_{122} &= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{12} + \sum_{\nu=1}^{n} \Gamma^1_{1\nu}\Gamma^n_{22} - \Gamma^1_{2\nu}\Gamma^\nu_{12} \\ &= -\partial_t \left(\frac{\dot\gamma_1(t)}{\gamma_1(t)} \right) + \frac{\dot\gamma_1(t)}{\gamma_1(t)} \cdot \frac{\dot\gamma_1(t)\ddot\gamma_1(t) + \dot\gamma_2(t)\ddot\gamma_2(t)}{|\dot\gamma(t)|^2} - \frac{\dot\gamma_1(t)^2}{\gamma_1(t)^2} \\ &= \frac{-\dot\gamma_2(t)^2\ddot\gamma_1(t) + \dot\gamma_1(t)\dot\gamma_2(t)\ddot\gamma_2(t)}{\gamma_1(t)(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)^2} \end{aligned}$$

b) (i) The cylinder is a surface of revolution for the constant curve $\gamma(t) = (1, 0)$. It follows from the formula in part a) that $K(p) = 0$ at point of the cylinder, since $\dot\gamma = 0$.

(ii) The sphere (with the north and south pole removed) is a surface of revolution for the curve $\gamma(t) = (\cos(t), \sin(t))$ with $t \in (-\pi/2, \pi/2)$. This satisfies

$$\dot\gamma(t) = (-\sin(t), \cos(t)), \qquad \ddot\gamma(t) = (-\cos(t), -\sin(t))$$

and the formula in a) yields

$$K(\psi(s,t)) = 1$$

where $\psi(s,t) = (\cos(t)\cos(s), \cos(t)\sin(s), \cos(t))$. It follows from the symmetry of S^2 that the Gauss curvature on the two poles is also 1.

(iii) The torus is a surface of revolution for the curve $\gamma(t) = (R + a\cos(t), a\sin(t))$. This satisfies

$$\dot\gamma(t) = (-a\sin(t), a\cos(t)), \qquad \ddot\gamma(t) = (-a\cos(t), -a\sin(t))$$

and the formula from a) yields

$$K(\psi(s,t)) = \frac{\cos(t)}{a(R + a\cos(t))}$$

with $\psi(s,t) = ((a\cos(t) + R)\cos(s), (a\cos(t) + R)\sin(s), a\sin(t))$.

c) For the following calculations recall that $\sinh^2(t) + 1 = \cosh^2(t)$. The derivatives of the tractrix are given by

$$\dot\gamma(t) = \left(\frac{-\sinh(t)}{\cosh^2(t)}, \frac{\sinh^2(t)}{\cosh^2(t)} \right)$$

$$\ddot\gamma(t) = \left(\frac{-\cosh^2(t) + 2\sinh^2(t)}{\cosh^3(t)}, \frac{2\sinh(t)\cosh^2(t) - 2\sinh^3(t)}{\cosh^3(t)} \right)$$

This yields

$$\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2 = \frac{\sinh^2(t) + \sinh^4(t)}{\cosh^4(t)} = \frac{\sinh^2(t)}{\cosh^2(t)} = \tanh^2(t)$$

and hence

$$\gamma_1(t)(\dot\gamma_1(t)^2+\dot\gamma_2(t)^2)^2=\frac{1}{\cosh(t)}\tanh^4(t).$$

For the nominator, we get

$$\begin{aligned}
&-\dot\gamma_2(t)^2\ddot\gamma_1(t)+\dot\gamma_1(t)\dot\gamma_2(t)\ddot\gamma_2(t)\\
&=-\tanh^4(t)\frac{-\cosh^2(t)+2\sinh^2(t)}{\cosh^3(t)}+\frac{-\tanh^3(t)}{\cosh(t)}\frac{2\sinh(t)\cosh^2(t)-2\sinh^3}{\cosh^3(t)}\\
&=-\tanh^4(t)\left[\frac{-\cosh^2(t)+2\sinh^2(t)}{\cosh^3(t)}+\frac{2\cosh^2(t)-2\sinh^3(t)}{\cosh^3(t)}\right]\\
&=-\tanh^4(t)\frac{1}{\cosh(t)}
\end{aligned}$$

It follows from part a) that the curvature of Beltrami's pseudosphere Σ is given by

$$K(\psi(s,t))=-1.$$

Note that Σ is singular along the circle $t=0$ and the curvature only well-defined for $t\neq 0$.

3. We call a manifold $M\subset\mathbb{R}^n$ **flat** if the Riemann curvature tensor R vanishes everywhere.

a) Prove that every 1− dimensional manifold is flat.

b) Let $M_i\subset\mathbb{R}^{n_i}$, $i=1,2$ be flat manifolds. Prove that $M=M_1\times M_2$ is flat.

c) Prove that a cylinder $Z=S^1\times\mathbb{R}\subset\mathbb{R}^3$ and the tori $T^n=S^1\times\cdots\times S^1\subset\mathbb{C}^n$ are flat.

Solution:

a) Let M be 1−dimensional manifold. For every $p\in M$ the Riemann curvature tensor

$$R_p:T_pM\times T_pM\to\mathrm{End}(T_pM)$$

is an anti-symmetric bilinear map satisfying $R_p(u,v)=-R_p(v,u)$ for all $u,v\in M$. With $u=v$ this yields $R_p(v,v)=0$ and more generally for $\mu,\lambda\in\mathbb{R}$ we have

$$R_p(\mu v,\lambda v)=\mu\lambda R_p(v,v)=0.$$

Since M is one-dimensional, we have $T_pM=\mathbb{R}v$ for any non-zero element $v\in T_pM$. Thus $R_p=0$.

b) Recall that the tangent space at $(p_1,p_2)\in M_1\times M_2$ is given as

$$T_{(p_1,p_2)}(M_1\times M_2)=T_{p_1}M_1\times T_{p_2}M_2.$$

Consider vector field $X,Y\in\mathrm{Vect}(M_1\times M_2)$ of the shape

$$X(p_1,p_2)=(X_1(p_1),X_2(p_2)),\qquad Y(p_1,p_2)=(Y_1(p_1),Y_2(p_2)).$$

where $X_1,Y_1\in\mathrm{Vect}(M_1)$ and $X_2,Y_2\in\mathrm{Vect}(M_2)$. Then

$$\begin{aligned}
\nabla_XY(p_1,p_2)&=\Pi(p_1,p_2)dX(p_1,p_2)Y(p_1,p_2)\\
&=\Pi(p_1,p_2)\left(dX_1(p_1)Y_1(p_1),\,dX_2(p_2)Y_2(p_2)\right))\\
&=\Big(\Pi^{M_1}(p_1)dX_1(p_1)Y_1(p_1),\,\Pi^{M_2}(p_2)dX_2(p_2)Y_2(p_2)\Big)\\
&=(\nabla_{X_1}Y_1(p_1),\,\nabla_{X_2}Y_2(p_2))
\end{aligned}$$

Interchanging the roles of X and Y this yields

$$[X,Y](p_1,p_2) = (\nabla_Y X - \nabla_X Y)(p_1,p_2) = ([X_1,Y_1](p_1),[X_2,Y_2](p_2)).$$

Note that $\nabla_X Y$ and $[X,Y]$ are both vector fields on $M_1 \times M_2$ which have the same product structure as X, Y.

Now suppose that $Z = (Z_1, Z_2) \in \mathrm{Vect}(M)$ is a third vector field having the same product structure as X and Y. We use the formula for the covariant derivatives and Lie bracket from above to calculate the curvature tensor at $p = (p_1, p_2)$:

$$\begin{aligned} R_p(X(p),Y(p))Z(p) &= \nabla_X \nabla_Y Z(p) - \nabla_Y \nabla_X Z(p) - \nabla_{[X,Y]} Z(p) \\ &= \Big(\nabla_{X_1}\nabla_{Y_1} Z_1(p_1) - \nabla_{Y_1}\nabla_{X_1} Z_1(p_1) - \nabla_{[X_1,Y_1]} Z_1(p_1), \\ &\qquad \nabla_{X_2}\nabla_{Y_2} Z_2(p_2) - \nabla_{Y_2}\nabla_{X_2} Z_2(p_2) - \nabla_{[X_2,Y_2]} Z_2(p_2)\Big) \\ &= \Big(R^{M_1}_{p_1}(X_1(p_1),Y_1(p_1))Z_1(p_1),\ R^{M_2}_{p_2}(X_2(p_2),Y_2(p_2))Z_2(p_2)\Big) \end{aligned}$$

Thus $M_1 \times M_2$ is flat if and only if M_1 and M_2 are flat.

c) As S^1 and $\mathbb{R}$ are 1-manifolds, a) tells us that these are flat. Thus it follows from b) that $Z = S^1 \times \mathbb{R}$ and $T^n = T^{n-1} \times S^1$ are flat (the latter following by induction over n).

4. **a)** For a smooth curve $\gamma : \mathbb{R} \to M$ and a normal vector field $Y \in \mathrm{Vect}^{\perp}(\gamma)$ define the covariant derivate by

$$\nabla^{\perp} Y(t) = (\mathbb{1} - \Pi(\gamma(t)))\dot{Y}(t).$$

Then $\nabla^{\perp} Y \in \mathrm{Vect}^{\perp}(\gamma)$ is again a normal vector field along γ. Show that

$$\dot{Y}(t) = \nabla^{\perp} Y(t) - h_{\gamma(t)}(\dot{\gamma}(t))^* Y(t)$$

b) The curvature tensor of $TM^{\perp}$ is a collection of linear map $R^{\perp}_p : T_pM \times T_pM \to \mathcal{L}(T_pM^{\perp}, T_pM^{\perp})$ defined by

$$R^{\perp}(\partial_s \gamma, \partial_t \gamma) Y = \nabla^{\perp}_s \nabla^{\perp}_t Y - \nabla^{\perp}_t \nabla^{\perp}_s Y$$

where $\gamma : \mathbb{R}^2 \to M$ denotes now a two parameter family and $Y \in \mathrm{Vect}^{\perp}(\gamma)$. Show that

$$R^{\perp}_p(u,v) = h_p(u)h_p(v)^* - h_p(v)h_p(u)^*$$

for all $p \in M$ and $v, w \in T_pM$.

Hint: We have shown in exercise 5 of exercise sheet 5 that $h_p(v)^* : T_pM^{\perp} \to T_pM$ is given by $h_p(v)^*\xi = (d\Pi(p)v)\xi$.

Solution:

a) Differentiating the equation $Y(t) = (\mathbb{1} - \Pi(\gamma(t)))\, Y(t)$ yields

$$\dot{Y}(t) = (\mathbb{1} - \Pi(\gamma(t)))\, \dot{Y}(t) - (d\Pi(\gamma(t))\dot{\gamma}(t))\, Y(t)$$

It follows from exercise 5 of exercise sheet 5 that

$$(d\Pi(\gamma(t))\dot{\gamma}(t))\, Y(t) = h_{\gamma(t)}(\dot{\gamma}(t))^* Y(t)$$

and this yields the claimed formula.

b) We have

$$\nabla_s^\perp \nabla_t^\perp Y = (\mathbb{1} - \Pi(\gamma))\, \partial_s \nabla_t Y$$

where the inner term expands to

$$\partial_s \nabla_t Y = \partial_s \partial_t Y - \Pi(\gamma)\partial_s\partial_t Y - (d\Pi(\gamma)\partial_s\gamma)\partial_t Y$$

It follows from a) that

$$\partial_t Y = \nabla_t^\perp Y - h_\gamma(\partial_s\gamma)^* Y$$

where the first term is orthogonal and the second term is tangential to M. With this it follows

$$\begin{aligned}(d\Pi(\gamma)\partial_s\gamma)\partial_t Y &= (d\Pi(\gamma)\partial_s\gamma)\nabla_t^\perp Y - (d\Pi(\gamma)\partial_s\gamma)(h_\gamma(\partial_t\gamma)Y) \\ &= h_\gamma(\partial_s\gamma)^*\nabla_t^\perp Y - h_\gamma(\partial_s\gamma)h_\gamma(\partial_s\gamma)^* Y\end{aligned}$$

where we used again the formula for the adjoint of h obtained in exercise 5 of exercise sheet 5. In summary we get

$$\partial_s \nabla_t Y = \partial_s\partial_t Y - \Pi(\gamma)\partial_s\partial_t Y - h_\gamma(\partial_s\gamma)^*\nabla_t^\perp Y + h_\gamma(\partial_s\gamma)h_\gamma(\partial_t\gamma)^* Y.$$

Since the second partial derivatives commute, this yields

$$\begin{aligned}\partial_s\nabla_t Y - \partial_t\nabla_s Y = {} & h_\gamma(\partial_t\gamma)^*\nabla_s^\perp Y - h_\gamma(\partial_s\gamma)^*\nabla_t^\perp Y \\ & + h_\gamma(\partial_s\gamma)h_\gamma(\partial_t\gamma)^* Y - h_\gamma(\partial_t\gamma)h_\gamma(\partial_s\gamma)^* Y.\end{aligned}$$

The first two terms are tangential vectors and the last two are normal vectors. Hence

$$\begin{aligned}\nabla_s^\perp\nabla_t^\perp Y - \nabla_t^\perp\nabla_s^\perp Y &= (\mathbb{1} - \Pi(\gamma))\,[\partial_s\nabla_t Y - \partial_t\nabla_s Y] \\ &= h_\gamma(\partial_s\gamma)h_\gamma(\partial_t\gamma)^* Y - h_\gamma(\partial_t\gamma)h_\gamma(\partial_s\gamma)^* Y\end{aligned}$$

and this yields the formula for the curvature tensor $R^\perp$.

5. Prove that on a complete manifold, any Killing vector field is complete.

Hint: Start by proving that $|\dot\gamma|$ is constant along any flow line γ of X.

Solution: Let ϕ^t be the flow of X and take $p \in M$. Let $I(p)$ be the maximal interval of existence and suppose that $I(p) = (a, b)$ with $b < \infty$. First we claim that $\langle X \circ \gamma, X \circ \gamma\rangle$ is a constant function along $\gamma : I(p) \to M : t \mapsto \phi^t(p)$. Thus let $q = \phi^t(p)$, then we have

$$\langle X(q), X(q)\rangle = \langle X(\phi^t(p)), X(\phi^t(p))\rangle = \langle d\phi^t(p)X(p), d\phi^t(p)X(p)\rangle = \langle X(p), X(p)\rangle,$$

where the second equality follows from the defining property of the flow and the last equality follows by definition of a Killing field. Now we use completeness of M to extend γ on the right, thus leading to a contradiction to $b < \infty$. Indeed, take $t_i \in I$ a sequence converging to b. Then t_i is in particular Cauchy and we have the estimate for $t_i < t_j$

$$d(\gamma(t_i), \gamma(t_j)) \le \mathrm{L}(\gamma|_{[t_i,t_j]}) = |X(p)||t_j - t_i|.$$

Thus $\{\gamma(t_i)\}_i$ is a Cauchy sequence and by completeness of M, it converges to $\gamma(b) := \lim_{t\to b}\gamma(t)$. But then $b \in I(p)$ and we can extend γ to $I(p) \cup (b, b + \epsilon)$ which is a contradiction to the maximality of $I(p)$.

6. Let X, Y be Killing vector fields on a complete manifold M.

a) Prove that $X + \lambda Y$ is again a Killing vector field for all $\lambda \in \mathbb{R}$.

b) Prove that if there is $p \in M$ such that $X(p) = Y(p)$ and $\nabla_v X = \nabla_v Y$ for all $v \in T_pM$, then $X = Y$.

c) Give an upper bound on the dimension of the vector space of Killing fields.

d) Write down all Killing fields on $\mathbb{R}^3$.

Hint: For a), prove that a vector field verifying the relation $\langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0$, for all $p \in M$ and $v, w \in T_pM$ is a Killing field. For b), use a), to conclude that it is enough to prove that a Killing field with $X(p) = \nabla X(p) = 0$ has to be zero. Prove that $d\phi^t(p) = \mathbb{1}$ for all $t \in \mathbb{R}$ for the flow of X. Conclude by using the uniqueness of isometries. For c), bunching together all the $\nabla_v X$ will give you a linear map with a special property, so that you get a better bound than $m + m^2$.

Solution:

a) We assume that $\langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0$ holds for all $p \in M$ and $v, w \in T_pM$. We will prove that X is Killing.

Take $p \in M$ and $v, w \in T_pM$. Denote by $w_s := d\phi^s(p)w$ and $v_s := d\phi^s(p)v$ for $s \in \mathbb{R}$. Then we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=s} \langle v_t, w_t\rangle &= \left.\frac{d}{dt}\right|_{t=0} \langle d\phi^t(\phi^s(p))v_s, d\phi^t(\phi^s(p))w_s\rangle \\ &= \langle dX(\phi^s(p))v_s, w_s\rangle + \langle v_s, dX(\phi^s(p))w_s\rangle \\ &= \langle \nabla_{v_s} X(\phi^s(p)), w_s\rangle + \langle v_s, \nabla_{w_s} X(\phi^s(p))\rangle = 0\end{aligned}$$

where the penultimate equation uses that w_s and v_s tangential vectors and the last equality follows from the assumption. Thus

$$\langle d\phi^t(p)v, d\phi^t(p)w\rangle = \langle d\phi^0(p)v, d\phi^0(p)w\rangle = \langle v, w\rangle,$$

and so X is Killing. Therefore, from X, Y Killing, we get

$$\langle \nabla_v(X+\lambda Y), w\rangle + \langle v, \nabla_w(X+\lambda Y)\rangle = \langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle + \lambda(\langle \nabla_v Y, w\rangle + \langle v, \nabla_w Y\rangle) = 0,$$

for all $p \in M$ and $v, w \in T_pM$. Therefore, $X + \lambda Y$ is also Killing.

b) By a), we get that given X, Y Killing fields, then $X - Y$ is again a Killing field. So we are left with showing that a Killing field X, satisfying for some $p \in M$, $X(p) = 0$ and $\nabla_Y X(p) = 0$ for every vector field Y, is zero.

Let X be such a Killing field. Then we have the following

(i) The flow ϕ^t leaves p unchanged, i.e $\phi^t(p) = p$ for all t.

(ii) By the torsion free property, we conclude for all $Y \in \mathrm{Vect}(M)$

$$[X, Y](p) = \nabla_{Y(p)} X(p) - \nabla_{X(p)} Y(p) = 0.$$

So we can calculate using $(\phi^s)_* Y(p) = d\phi^s(p)Y(p)$ for all $s \in \mathbb{R}$ and $Y \in \mathrm{Vect}(M)$

$$\left.\frac{d}{dt}\right|_{t=s} d\phi^t(p)Y(p) = d\phi^s(p) \left.\frac{d}{dt}\right|_{t=0} (\phi^t)_* Y(p) = d\phi^s(p)[X, Y](p) = 0.$$

Now since $d\phi^0 = \mathbb{1}$, we conclude that $d\phi^t = \mathbb{1}$ for all $t \in \mathbb{R}$. By uniqueness of isometries and as $\phi^t(p) = p$ and $d\phi^t(p) = \mathbb{1}$, $\phi^t = \mathrm{id}$ and so $X = 0$.

c) Take any point $p \in M$ and $\{b_i\}_{i=1,\dots,m} \subset T_pM$ a basis of the tangent space. Denote by $\mathcal{K}(M)$ the vector space of Killing fields and define the linear map

$$i : \mathcal{K}(M) \to T_pM \oplus \mathrm{Asym}(T_pM) : X \mapsto (X(p), m(X,p)),$$

where $m(X,p) : T_pM \to T_pM$ is defined as $m(X,p)v := \nabla_v X(p)$ and $\mathrm{Asym}(T_pM)$ denotes the skew-symmetric endomorphism of T_pM. By a), the map i is well-defined as

$$\langle m(X,p)v, w\rangle + \langle v, m(X,p)w\rangle = \langle \nabla_v X, w\rangle + \langle v, \nabla_w X\rangle = 0.$$

By b), this linear map i is injective, so

$$\dim \mathcal{K}(M) \leq m + \frac{m(m-1)}{2}.$$

d) For $m = 3$, the upper bound from c) is 6. We can easily generate families of translations by the constant vector fields $X_v(x) = v$ and rotations around 0 are generated by $Y_v(x) = v \times x$. Thus by a), $\mathrm{span}\{X_v, Y_v : v \in \mathbb{R}^3\} \subset \mathcal{K}(\mathbb{R}^3)$ and for dimensional reasons

$$\mathrm{span}\{X_v, Y_v : v \in \mathbb{R}^3\} = \mathcal{K}(\mathbb{R}^3).$$