Solution 12

1. Prove that S^n is simply connected for $n \geq 2$.

Hint: You may use that the image of a smooth curve $\gamma : [a, b] \to S^n$ has measure zero. This is a special case of Sard's theorem which says that for any smooth map $f : M \to N$ the set of regular values has full measure. This theorem will be a major result from the second semester.

Solution: Let $\gamma_i : [a, b] \to S^n$ be a path for $i = 0, 1$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. By Sard's theorem there is a point $p \in S^n$ which is a regular value for γ_0 and γ_1. This means that $d\gamma_i(t)$ has rank n for all $t \in \gamma_i^{-1}(p)$ for $i = 0, 1$. As the rank of $d\gamma_i(t)$ can be at most 1 and $n \geq 2$, this simply means that p is not in the image of the two curves γ_0 and γ_1. Thus take the stereographic projection φ_p of S^n from p to $\mathbb{R}^n$. Then take the homotopy $\gamma : [0, 1] \times [a, b] \to S^n$ given by $\gamma_\lambda(t) = \varphi_p^{-1}((1 - \lambda)\varphi_p(\gamma_0(t)) + \lambda\varphi_p(\gamma_1(t)))$ from γ_0 to γ_1 with fixed endpoints. Thus S^n is simply connected.

2. **(Developable hypersurfaces)** Let $n = m + 1$ and let $E(t)$ be a one-parameter family of hyperplanes in $\mathbb{R}^n$. Then there is a smooth map $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$E(t) = u(t)^\perp, \qquad |u(t)| = 1$$

for every t. We assume that $\dot{u}(t) \neq 0$ for every t so that $u(t)$ and $\dot{u}(t)$ are linearly independent.

a) Show that

$$L(t) := u(t)^\perp \cap \dot{u}(t)^\perp = \lim_{s \to t} E(s) \cap E(t).$$

Thus $L(t)$ is a linear subspace of dimension $m - 1$.

b) Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that

$$\langle \dot{\gamma}(t), u(t) \rangle = 0, \qquad \langle \dot{\gamma}(t), \dot{u}(t) \rangle \neq 0.$$

for all t. This means that $\dot{\gamma}(t) \in E(t)$ and $\dot{\gamma}(t) \notin L(t)$; thus $E(t)$ is spanned by $L(t)$ and $\dot{\gamma}(t)$. For $t \in \mathbb{R}$ and $\epsilon > 0$ define

$$L(t)_\epsilon = \{v \in L(t) \mid |v| < \epsilon\}.$$

Let $I \subset \mathbb{R}$ be a bounded open interval such that the restriction of γ to the closure of I is injective. Prove that, for $\epsilon > 0$ sufficiently small, the set

$$M_0 := \bigcup_{t \in I} (\gamma(t) + L(t)_\epsilon).$$

is a smooth manifold of dimension $m = n - 1$ and the spaces $L(t)_\epsilon$ are all disjoint for different t. A manifold which arises this way is called **developable**.

c) Show that the tangent spaces of M_0 are the original subspaces $E(t)$, i.e.

$$T_p M_0 = E(t) \quad \text{for} \quad p \in \gamma(t) + L(t)_\epsilon.$$

One therefore calls M_0 the *envelope* of the hyperplane $\gamma(t) + E(t)$.

d) Show that M_0 is flat.

e) If (Φ, γ, γ') is a development of M_0 along $\mathbb{R}^m$, show that the map $\phi : M_0 \to \mathbb{R}^m$, defined by

$$\phi(\gamma(t) + v) = \gamma'(t) + \Phi(t)v$$

for $v \in L(t)_\epsilon$ is an isometric immersion onto an open set $M_0' \subset \mathbb{R}^m$. Thus a development *unrolls* M_0 onto the Euclidean space $\mathbb{R}^m$. When $n = 3$ and $m = 2$ one can visualize M_0 as a twisted sheet of paper.

Hint: Part a) is not as innocent as it seems; skip this part if you get stuck. For b): Solutions of the ODE $\dot{X}(t) + \frac{\langle X(t), \ddot{u}(t) \rangle}{|\dot{u}(t)|^2} \dot{u}(t) = 0$ remain in $L(t)$. This can be used to construct a smooth family of orthonormal bases $X_1(t), \ldots, X_{m-1}(t)$ along $L(t)$. Use these and the implicit function theorem to construct charts for M_0. For d): Use the Gauss-Codazzi formula. For e): Recall that parallel transport intertwines along developements.

Solution:

a) We show first $\lim_{s\to t} E(s)\cap E(t) \subset u(t)^\perp \cap \dot u(t)^\perp$. Let $v(s) \in E(s)\cap E(t)$ be a continuous curve satisfying defined for all s with $|s-t|$ sufficiently small. Then $\langle v(s), u(t)\rangle = 0$ and $\langle v(s), u(s)\rangle = 0$ for all s. Differentiating the second condition at $s = t$ yields

$$0 = \partial_s|_{s=t}\langle v(s), u(t)\rangle + \partial_s|_{s=t}\langle v(t), u(s)\rangle = \langle v(t), \dot u(t)\rangle.$$

Hence $v(t) \in \dot u(t)^\perp$ and the second condition $v(t) \in E(t)$ is obvious.

We show next $u(t)^\perp \cap \dot u(t)^\perp \subset \lim_{s\to t} E(s) \cap E(t)$. Let $v \in u(t)^\perp \cap \dot u(t)^\perp$ be given. We want to extend v to a continuous path $v(s)$ satisfying $v(s) \in E(t) \cap E(s)$ and $v(t) = v$. For this we make the ansatz

$$v(s) = v - \alpha u(s) - \beta u(t).$$

The coefficients are determined by the equations $\langle v(s), u(s)\rangle = 0$, $\langle v(s), u(t)\rangle = 0$ and a short calculation shows that

$$v(s) = v - \frac{\langle v, u(s)\rangle}{1 - \langle u(s), u(t)\rangle^2} u(s) + \langle u(s), u(t)\rangle \frac{\langle v, u(s)}{1 - \langle u(s), u(t)\rangle^2} u(t)$$

The limit of $v(s)$ as $s \to t$ does not necessarily exists. We can fix this by considering the following rescaling first

$$v'(s) = \left(1 - \langle u(s), u(t)\rangle^2\right) v - \langle v, u(s)\rangle \left(u(s) - \langle u(s), u(t)\rangle u(t)\right)$$

Differentiating the assumptions $|u| \equiv 1$ yields

$$\langle \dot u(t), u(t)\rangle = 0, \qquad \langle \ddot u(t), u(t)\rangle + |\dot u(t)|^2 = 0.$$

Together with the assumption $\langle v, u(t)\rangle = 0$ and $\langle v, \dot u(t)\rangle = 0$ a short calculation reveals

$$v'(t) = 0, \qquad \dot v'(t) = 0, \qquad \ddot v'(t) = -2\langle \ddot u(t), u(t)\rangle v = 2|\dot u(t)|^2 v$$

Hence

$$v''(s) := \frac{v'(s)}{2|\dot u(t)|^2 (s-t)^2}$$

satisfies $v''(s) \in E(s) \cap E(t)$ and $\lim_{s\to t} v''(s) = v$.

b) Let $t_0 \in \mathbb{R}$ and choose an orthonormal basis $e_1, \cdots, e_{m-1}$ of $L(t_0)$. Define $X_i(t)$ by the equation

$$\dot X_i(t) + \frac{\langle X_i(t), \ddot u(t)\rangle}{|\dot u(t)|^2} \dot u(t) = 0., \qquad X_i(t_0) = e_i \tag{1}$$

for $i = 1, \cdots, m-1$. We claim that $X_1(t), \ldots, X_{m-1}(t)$ form and orthonormal basis of $L(t)$ for all t, i.e.

$$\langle X_i(t), \dot u(t)\rangle = 0, \qquad \langle X_i(t), u(t)\rangle = 0, \qquad \langle X_i(t), X_j(t)\rangle = \delta_{ij}.$$

For $t = t_0$ these equations are clearly satisfied, since $X_i(t_0) = e_i$ form an orthonormal basis of $L(t_0)$. It follows from (1) that

$$\begin{aligned}\partial_t\langle X_i(t), \dot u(t)\rangle &= \langle \dot X_i(t), \dot u(t)\rangle + \langle X_i(t), \ddot u(t)\rangle \\ &= -\langle X_i(t), \ddot u(t)\rangle + \langle X_i(t), \ddot u(t)\rangle = 0\end{aligned}$$

which yields $\langle X_i(t), \dot u(t)\rangle = 0$ for all $t \in I$. It now follows from equation and (1) that

$$\begin{aligned}\partial_t \langle X_i(t), u(t)\rangle &= \langle \dot X_i(t), u(t)\rangle + \langle X_i(t), \dot u(t)\rangle \\ &= -\frac{\langle \dot u(t), u(t)\rangle\langle X_i(t), \ddot u(t)\rangle}{|\dot u(t)|^2} + \langle X_i(t), \dot u(t)\rangle = 0\end{aligned}$$

and thus $\langle X_i(t), u(t)\rangle = 0$ for all $t \in I$. Finally, we deduce from (1) that

$$\begin{aligned}\partial_t \langle X_i(t), X_j(t)\rangle &= \langle \dot X_i(t), X_j(t)\rangle + \langle X_i(t), \dot X_j(t)\rangle \\ &= -\frac{\langle \dot u(t), X_j(t)\rangle\langle X_i(t), \ddot u(t)\rangle}{|\dot u(t)|^2} - \frac{\langle \dot u(t), X_j(t)\rangle\langle X_i(t), \ddot u(t)\rangle}{|\dot u(t)|^2} = 0\end{aligned}$$

and thus $\langle X_i(t), X_j(t)\rangle = \delta_{ij}$ for all $t \in I$.

We define the local parametrization $\psi : \mathbb{R}^m \to \mathbb{R}^n$ by

$$\psi(t, x_1, x_2, \cdots, x_{m-1}) = \gamma(t) + \sum_{i=1}^{m-1} x_i X_i(t).$$

The derivative of ψ at the point $(t_0, 0, \cdots, 0)$ is given by

$$d\psi(t_0, 0, \cdots, 0)(\hat t, \hat x_1, \ldots, \hat x_{m-1}) = \dot\gamma(t)\hat t + \sum_{i=1}^{m-1} \hat x_i X_i(t).$$

This is clearly injective. It follows from the implicit function theorem that there exists an open neighborhood $(t_0, 0) \in U_{t_0} \subset \mathbb{R}^m$ such that the restriction $\psi|_U : U \to \psi(U)$ is a diffeomorphism (onto its image). Now take a smaller open set $J(t_0) \times B_{\epsilon_{t_0}} \subset U$ and consider the restriction

$$\psi^{t_0} : J(t_0) \times B_{\epsilon_{t_0}}(0) \subset \psi(J(t_0) \times B_{\epsilon_{t_0}})$$

We may apply this construction to every $t_0 \in \overline{I}$. Since $\overline{I}$ is compact, there exists finitely many times $t_1, \ldots, t_k$ such that $I \subset J(t_1) \cup \ldots \cup J(t_k)$. Define $\epsilon := \min\{\epsilon_{t_1}, \ldots, \epsilon_{t_k}\}$. Then each of the maps $\psi^{t_1}, \ldots, \psi^{t_k}$ restricts to a chart of

$$M_0 := \bigcup_{t\in I} (\gamma(t) + L(t)_\epsilon) = \{\gamma(t) + v \mid v \in L(t),\ |v| < \epsilon\}.$$

These charts cover M_0 and hence this is an m-dimensional manifold.

Note that for $\epsilon > 0$ sufficently small, we may assume in addition that $L(t)_\epsilon \cap L(t')_\epsilon = \emptyset$ for $t, t' \in I$ and $t \neq t'$. When they belong to the same chart, i.e. $t, t' \in J(t_i)$ for some i, then this follows directly from injectivity of the chart ψ^{t_i}. When they are not contained in a single chart then there exists a universal constant $\delta > 0$ such that $|\gamma(t) - \gamma(t')| > \delta$. We can now guarantee injectivity by choosing $\epsilon < \delta/2$

c) For $p \in M_0$ choose $x_1, \ldots, x_{m-1} \in \mathbb{R}$ such that $p = \gamma(t) + \sum_{i=1}^{m-1} x_i X_i(t)$. Differentiating the curve $s \mapsto p + X_i(t)s$ shows that $X_i(t) \in T_pM_0$. Differentiating the curve $s \mapsto \gamma(s) + \sum_{i=1}^{m-1} x_i X_i(s)$ yields

$$\tau := \dot\gamma(t) + \sum_{i=1}^{m-1} x_i \dot X_i(t) \in T_pM.$$

Note that this cannot be contained in $L(t)$: Otherwise, the derivative of the corresponding chart $d\psi(t, x_1, \ldots, x_{m-1})$ would have a kernel which is impossible for a diffeomorphism. We could guarantee this in part b) by chosing $\epsilon > 0$ sufficiently small. Using (1) we can rewrite

$$\tau = \dot\gamma(t) - \sum_{i=1}^{m-1} x_i \frac{\langle X_i(t), \ddot u(t)\rangle}{|\dot u(t)|^2} \dot u(t)$$

and this implies $\tau \in E(t)$. It follows that $\mathrm{Span}(\tau, X_1(t), \ldots, X_{m-1}) = E(t) \subset T_pM_0$. We know from part c) that M_0 is a m-dimensional manifold and thus we must have equality for dimensional reasons. This shows $E(t) = T_pM_0$.

d) The map $u : M_0 \to S^m$ is by definition an unit normal vector field. Hence the orthogonal projections onto the tangent spaces are

$$\Pi(p) = \mathbb{1} - u(t)u(t)^\top \qquad \text{for } p \in \gamma(t) + L(t)_\epsilon.$$

We take the same basis $\tau, X_1(t), \ldots, X_{m-1}(t)$ of T_pM_0 as in part c). For this basis it follows

$$h_p(X_i) = d\Pi(p)X_i = 0, \qquad h_p(\tau) = d\Pi(p)\tau = -\dot u(t)u(t)^\top - u(t)\dot u(t)^\top.$$

Since the dual of the zero map is the zero map, we have also $h_p(X_i)^* = 0$. In particular $h_p(v)^*h_p(w) = 0$ for all basis vectors $v, w \in \{\tau, X_1(t), \ldots, X_{m-1}(t)\}$ unless $v = \tau = w$. The Gauss-Codazzi equation now yields that

$$R_p(v, w) = h_p(v)^*h_p(w) - h_p(w)^*h_p(v) = 0$$

for all basis vectors $v, w \in \{\tau, X_1(t), \ldots, X_{m-1}(t)\}$. By linearity, this suffices to conclude that $R_p = 0$ vanishes.

e) Let $p \in M_0$ and $v(s) \in L(s)_\epsilon$ be a smooth curve defined for $|s - t|$ sufficiently small such that $p = \gamma(t) + v(t)$. Since $\dot\gamma(s) \in E(s) = T_{\gamma(s)+v(s)}M_0$, it follows that

$$\nabla_s v(s) = \dot v(s) \qquad \text{for all } s.$$

The covariant derivatives on $\mathbb{R}^m$ (which is trivial) intertwines with the covariant derivative in M_0 along the developement. This yields

$$\partial_s\left(\Phi(s)v(s)\right) = \Phi(s)\nabla_s v(s) = \Phi(s)\dot v(s).$$

We also have that $\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$. This yields

$$\begin{aligned}
d\phi(p)[\dot\gamma(t) + \dot v(t)] &= \frac{d}{ds}\bigg|_{s=t} \phi(\gamma(s) + v(s)) \\
&= \frac{d}{ds}\bigg|_{s=t} (\gamma'(s) + \Phi(s)v(s)) \\
&= \dot\gamma'(t) + \partial_s(\Phi(s)v(s))|_{s=t} \\
&= \Phi(t)\dot\gamma(t) + \Phi(t)\dot v(t) \\
&= \Phi(t)(\dot\gamma(t) + \dot v(t))
\end{aligned}$$

Since every tangent vector in T_pM_0 can represented in the form $\dot\gamma(t) + \dot v(t)$, it follows that $d\phi(p) = \Phi(t) : E(t) \to \mathbb{R}^m$ and this is an orthogonal isomorphism. Hence ϕ is an isometry.

3. Prove that each of the following is a developable surface in $\mathbb{R}^3$.

a) A cone on an (embedded) curve $\Gamma \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ given by

$$M = \left\{ \lambda p + (1-\lambda) q \,\middle|\, \lambda < 1,\ q \in \Gamma \right\}, \quad p \in \mathbb{R}^3 \setminus \left(\mathbb{R}^2 \times \{0\}\right).$$

b) A cylinder on an (embedded) plane curve, i.e.

$$M = \left\{ q + tv \,\middle|\, t \in \mathbb{R},\ q \in \Gamma \right\},$$

where Γ are as in (i) and v is a fixed vector in $\mathbb{R}^3 \setminus (\mathbb{R}^2 \times \{0\})$. (This is the cone with a point p at infinity).

c) The tangent developable to a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$

$$M := \left\{ \gamma(t) + s\dot{\gamma}(t) \,\middle|\, |t - t_0| < \epsilon,\ \ 0 < s < \epsilon \right\},$$

where $\dot{\gamma}(t_0)$ and $\ddot{\gamma}(t_0)$ are linearly independant and $\epsilon > 0$ is sufficiently small.

Solution:

a) Suppose that Γ is parametrised by arc length, i.e. by a smooth injective curve $\gamma : I \to \Gamma \subset \mathbb{R}^2$ with $|\dot{\gamma}| = 1$. Let $u(t) = \dot{\gamma}(t) \times \frac{\gamma(t)-p}{|\gamma(t)-p|}$. Then $u(t)$ is a unit vector and $u(t)^\perp = \mathrm{span}\{\dot{\gamma}(t), \gamma(t) - p\}$. Differentiating $u(t)$ we get

$$\dot{u}(t) = \ddot{\gamma}(t) \times \frac{\gamma(t) - p}{|\gamma(t) - p|} + \frac{\langle \dot{\gamma}(t), \gamma(t) - p \rangle}{|\gamma(t) - p|^2} \dot{\gamma}(t) \times \frac{\gamma(t) - p}{|\gamma(t) - p|}.$$

So we get $u(t)^\perp \cap \dot{u}(t)^\perp = L(t) = \mathrm{span}\{\gamma(t) - p\}$. Therefore for $t \in I$ and $\lambda < 1$

$$M \ni \lambda\gamma(t) + (1-\lambda)p = \gamma(t) + (1-\lambda)(p - \gamma(t)) = \gamma(t) + v, \quad v \in L(t).$$

So M is developable.

b) M is a smooth manifold. Suppose as in a) that Γ is parametrised by arc length. Let $u(t) = v \times \dot{\gamma}(t)$. Then $u(t)$ is a unit vector and

$$u(t)^\perp \cap \dot{u}(t)^\perp = \mathrm{span}\{v\}.$$

Hence we get $M = M_0 = \{\gamma(t) + \mathbb{R}v \mid t \in I\}$.

c) Let $u(t) = \frac{\dot{\gamma}(t) \times \ddot{\gamma}(t)}{|\dot{\gamma}(t) \times \ddot{\gamma}(t)|}$. Then $u(t)$ is a unit vector as long as $|t - t_0| < \epsilon$ for ϵ sufficiently small, because linear independence is an open condition. Assume γ to be parametrised by arc length (this does not change the set M). Then $\langle \dot{\gamma}, \ddot{\gamma} \rangle = 0$ and so $|\dot{\gamma}(t) \times \ddot{\gamma}(t)| = |\ddot{\gamma}|$. Thus we can check

$$\dot{u}(t) = \dot{\gamma}(t) \times \frac{\dddot{\gamma}(t)}{|\ddot{\gamma}(t)|} - \dot{\gamma}(t) \times \ddot{\gamma}(t) \frac{\langle \ddot{\gamma}(t), \dddot{\gamma}(t) \rangle}{|\ddot{\gamma}(t)|^3} \qquad \text{and so}$$

$$L(t) = u(t)^\perp \cap \dot{u}(t)^\perp = \mathrm{span}\{\dot{\gamma}(t)\}.$$

So $M = M_0$ is developable.

4. For $b \geq a > 0$ and $c \geq 0$ define

$$M_{a,b,c} := \{(u, v, w) \in \mathbb{C}^3 \ :\ |u| = a, |v| = b, w = cuv\}.$$

a) Prove that $M_{a,b,c}$ is diffeomorphic to the standard torus $S^1 \times S^1$. Also prove that $M_{a,b,c}$ is flat.

b) Show that the natural map $F : M_{a,b,c} \to M_{a',b',c'}$ defined by

$$(u, v, w) \mapsto (u', v', w') := \left(\frac{ua'}{a}, \frac{vb'}{b}, \frac{c'ua'vb'}{ab}\right)$$

is an isometry, exactly if $(a, b, c) = (a', b', c')$.

c) (**Challenge**) When are two flat tori $M_{a,b,c}$ and $M_{a',b',c'}$ isometric?

Solution:

a) Let us define a map

$$\Phi : S^1 \times S^1 \to M_{a,b,c} : (e^{i\theta}, e^{i\varphi}) \mapsto (ae^{i\theta}, be^{i\varphi}, abce^{i(\theta+\varphi)})$$

and its inverse

$$\Phi^{-1} : M_{a,b,c} \to S^1 \times S^1 : (u, v, w) \to (\frac{u}{a}, \frac{v}{b}).$$

Both are smooth, and bijective, so Φ is a diffeomorphism. Take a point $p \in M_{a,b,c}$ and take a point $q \in S^1 \times S^1$ with $\Phi(q) = p$. Take an connected open neighbourhood $U := I \times J$ of q where neither I nor J are the full S^1. Then we can look at the parametrisation ψ on $\Omega \subset \mathbb{R}^2$ such that

$$(\theta, \varphi) \mapsto \Phi(e^{i\theta}, e^{i\varphi}) \quad (e^{i\theta}, e^{i\varphi}) \in I \times J.$$

In these coordinates, we can look at the metric g coming from $M_{a,b,c}$. Then we have that

$$g_{\theta\theta} = a^2 + a^2b^2c^2, g_{\varphi\varphi} = b^2 + a^2b^2c^2, g_{\theta\varphi} = a^2b^2c^2.$$

As this metric is constant, we have a flat manifold. (All Christoffel symbols are all identically zero.)

b) This map is clearly a diffeomorphism. If we call $\hat{\theta} = \frac{\partial\psi}{\partial\theta}$ and $\hat{\varphi} = \frac{\partial\psi}{\partial\varphi}$ the tangent vectors at a point $p = (ae^{it}, be^{is}, w) \in M_{a,b,c}$, then

$$dF(p)\hat{\theta} = (ie^{it}a', 0, c'a'b'ie^{i(s+t)}), \quad dF(p)\hat{\varphi} = (0, ie^{is}, c'a'b'ie^{i(s+t)})).$$

So for F to be an isometry, we need to require that

$$a^2 + a^2b^2c^2 = \langle\hat{\theta}, \hat{\theta}\rangle = \langle dF(p)\hat{\theta}, dF(p)\hat{\theta}\rangle = (a')^2 + (a')^2(b')^2(c')^2.$$

Similarly, we need $abc = a'b'c'$ and $b^2 + (abc)^2 = (b')^2 + (a'b'c')^2$. Thus $(a, b, c) = (a', b', c')$ for F to be an isometry.

c) Consider the following two geodesics in $M_{a,b,c}$:

$$\gamma_1 : [0, 1] \to M_{a,b,c}, \qquad \gamma_1(t) = (a, be^{2\pi \mathbf{i}t}, abce^{2\pi \mathbf{i}t})$$

$$\gamma_2 : [0, 1] \to M_{a,b,c}, \qquad \gamma_2(t) = (ae^{2\pi \mathbf{i}t}, b, abce^{2\pi \mathbf{i}t})$$

It follows from our calculation above, that these are indeed geodesics and moreover

$$\text{Length}(\gamma_1) = 2\pi(a^2b^2c^2 + b^2), \qquad \text{Length}(\gamma_2) = 2\pi(a^2b^2c^2 + a^2)$$

$$\langle\dot{\gamma}_1(0), \dot{\gamma}_2(0)\rangle = 4\pi^2a^2b^2c^2$$

Note that these three quantitees determine $b \geq a > 0$ and $c > 0$ uniquely.

We have the following intrinsic description of these curves. First fix a basepoint $p \in M_{a,b,c}$. There exists two unique closed geodesics $\gamma_1^{(p)}$ and $\gamma_2^{(p)}$ starting at p with the following properties: $\gamma_2^{(p)}$ is the shortest closed geodesic starting at p and $\gamma_1^{(p)}$ is the shortest closed geodesic starting at p which intersects $\gamma_1^{(p)}$ in precisely one point. We gave explicit formulae in the case $p = (a, b, abc)$ above and the uniqueness claim can be verified using the explicit description of the metric in the chart. The curves $\gamma_1^{(p)}$ and $\gamma_2^{(p)}$ for a general base point are obtained by translation in the chart, i.e.

$$\gamma_1^{(p)}(t) = (p_1, p_2 e^{2\pi \mathbf{i} t}, p_3 e^{2\pi \mathbf{i} t}), \qquad \gamma_2^{(p)}(t) = (p_1 e^{2\pi \mathbf{i} t}, p_2, p_3 e^{2\pi \mathbf{i} t}).$$

They have the same length and inner product at p as calculated before.

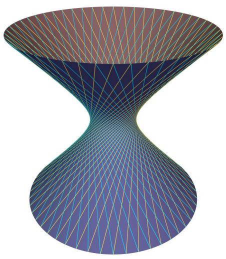
Since any isometry $f : M_{a,b,c} \to M_{a',b',c'}$ preserves the length of curves, it maps the geodesics $\gamma_i^{(p)}$ in $M_{a,b,c}$ to the corresponding geodesics $\gamma_i^{(f(p))}$ in $M_{a',b',c'}$. The length of these geodesics is preserved and determines the coefficients a, b, c and a', b', c'. It thus follows $(a, b, c) = (a', b', c')$.

5. A 2-dimensional submanifold $M \subset \mathbb{R}^3$ is called a ruled surface if there is a straight line in M through every point. Every developable surface is ruled, however, there are ruled surfaces that are not developable. An example is the elliptic hyperboloid of one sheet depicted below.

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \, : \, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \right\}.$$

Prove that this manifold has negative Gaussian curvature and there are two straight lines through every point in M.

Hint: For ease of calculation, assume that $a = b$ in the calculation of curvature. Last sheet had a formula for curvature of surfaces of revolution.



Solution: Let $p = (x_0, y_0, z_0) \in M$ and $m = (x_1, y_1, z_1) \in \mathbb{R}^3$ be given. Then $p + tm \in M$ for all $t \in \mathbb{R}$ if and only if

$$\frac{(x_0 + tx_1)^2}{a^2} + \frac{(y_0 + ty_1)^2}{b^2} = 1 + \frac{(z_0 + tz_1)^2}{c^2}$$

or equivalently

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + 2t\left(\frac{x_0 x_1}{a^2} + \frac{y_0 y_1}{b^2}\right) + t^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = 1 + \frac{z_0}{c^2} + 2t\frac{z_0 z_1}{c^2} + t^2\frac{z_1^2}{c^2}.$$

Since this equation has to hold for all $t \in \mathbb{R}$, the coefficients on both sides must agree. This yield three equations:

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 + \frac{z_0^2}{c^2}, \qquad \frac{x_0 x_1}{a^2} + \frac{y_0 y_1}{b^2} = \frac{z_0 z_1}{c^2}, \qquad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{z_1^2}{c^2}.$$

The first equation is automatically satisfied, since $p \in M$. The second and third equation can be solved most easily when p lies in the central ellipsoid, i.e. in the case $p = (a\cos(\theta), b\sin(\theta), 0)$. Then the second equation is solved by any multiple of $(x_1, y_1) = (b\sin(\theta), -a\cos(\theta))$ (and arbitrary z_1) and the third equations yields $z_1 = \pm c$ for this particular solution. It follows that there are exactly two lines passing through p which are contained in M. These are given by

$$\ell_\theta^\pm(t) = (a\cos(\theta), b\sin(\theta), 0) + t(b\sin(\theta), -a\cos(\theta), \pm c).$$

Now let $p \in M$ be a general point as above. For any line $p + tm$ contained in M which passes through p, we must have $z_1 \neq 0$, since $\{(x, y, z) \in M \,|\, z = z_0\}$ is bounded. Hence any such line passes for some t through the central ellipse and must agree (up to parametrization) with one of the lines $\ell_\theta^\pm(t)$. It follows from the formula that $t = \pm z_0/c$ and that $\cos(\theta), \sin(\theta)$ are uniquely determined by t and x_0, y_0. Therefore θ is also uniquely determined and there are two lines through p.

For the curvature, assume $a = b$. Then M is the surface of revolution of $\gamma(t) = (a\cosh(t), c\sinh(t))$ and so that we can apply the formula from last week.

$$\begin{aligned}
\kappa(\psi(s,t)) &= \frac{-\dot\gamma_2(t)^2\ddot\gamma_1(t) + \dot\gamma_1(t)\dot\gamma_2(t)\ddot\gamma_2(t)}{\gamma_1(t)(\dot\gamma_1(t)^2 + \dot\gamma_2(t)^2)^2} \\
&= \frac{-ac^2\cosh^3(t) + ac^2\cosh(t)\sinh^2(t)}{a\cosh(t)(a\cosh^2(t) + c\sinh^2(t))^2} \\
&= -c^2\frac{\cosh(t)}{(a\cosh^2(t) + c\sinh^2(t))^2} < 0.
\end{aligned}$$

6. Prove that the homotopy relation $\gamma_0 \sim \gamma_1$ is an equivalence relation on the space Ω_{pq} of curves with endpoints $p, q \in M$.

Solution: We check the axioms.

(i) **Symmetric** If $\gamma : [0,1]\times[a,b] \to M$ is a homotopy from γ_0 to γ_1 with fixed endpoints, then $\lambda \mapsto \gamma_{1-\lambda}$ is a homotopy from γ_1 to γ_1.

(ii) **Reflexive** For $\gamma_0 : [a,b] \to M$, we define the homotopy $\gamma : [0,1] \times [a,b] \to M : (t, \lambda) \mapsto \gamma_0(t)$ is a homotopy from γ_0 to γ_0.

(iii) **Transitivity** This is the only subtle point. Given homotopies γ from γ_0 to γ_1 and γ' from γ_1 to γ_2, we need a **smooth** homotopy from γ_0 to γ_2. This can be done using a smooth function $\rho : [0,1] \to [0,1]$ with the properties $\rho|_{[0,\epsilon)} \equiv 0$, $\rho|_{(1-\epsilon,1]} \equiv 1$ and image$(\rho) \subset [0,1]$ for some small $\epsilon > 0$ as follows. Define

$$\gamma''_\lambda(t) = \begin{cases} \gamma_{\rho(2\lambda)}(t), & \text{for } \lambda \in [0, \frac{1}{2}],\ t \in [a,b], \\ \gamma'_{\rho(2\lambda-1)}(t), & \text{for } \lambda \in [\frac{1}{2}, 1],\ t \in [a,b], \end{cases}$$

which is a smooth homotopy from γ_0 to γ_2.