Solution 13

1. **a)** Show that the curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

is multi-linear over $C^\infty(M,\mathbb{R})$.

b) Show that the covariant derivative

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W$$

is multi-linear over $C^\infty(M,\mathbb{R})$.

c) Deduce that the covariant derivative $(\nabla_X R)(Y,Z)W(p)$ depends only on the values $X(p), Y(p), Z(p), W(p) \in T_pM$ and not on the particular vector fields.

Hint: In a) you have to prove that $R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = f(R(X,Y)Z)$ holds for all $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M,\mathbb{R})$. For this recall that $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X(fY) = (\mathcal{L}_X f)Y + f\nabla_X Y$. Part b) goes similar. The claim in c) is in fact equivalent to multi-linearity over $C^\infty(M,\mathbb{R})$.

Solution:

a) We show $R(fX,Y)Z = fR(X,Y)Z$. First note that

$$[fX,Y] = \nabla_Y(fX) - \nabla_{fX}Y = (\mathcal{L}_Y f)X + f\nabla_Y X - f\nabla_X Y = (\mathcal{L}_Y f)X + f[X,Y]$$

and hence

$$\nabla_{[fX,Y]}Z = \nabla_{(\mathcal{L}_Y f)X + f[X,Y]}Z = (\mathcal{L}_Y f)\nabla_X Z + f\nabla_{[X,Y]}Z.$$

With this, we can verify

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z + \nabla_{[fX,Y]}Z \\ &= f\nabla_X\nabla_Y Z - (\mathcal{L}_Y f)\nabla_X Z - f\nabla_Y\nabla_X Z + (\mathcal{L}_Y f)\nabla_X Z + f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z \end{aligned}$$

The second equation $R(X,fY)Z = fR(X,Y)Z$ follows from a similar calculation. Alternatively it can be deduced from the first one via

$$R(X,fY)Z = -R(fY,X)Z = -fR(Y,X)Z = fR(X,Y)Z.$$

We verify the last equation $R(X,Y)(fZ) = fR(X,Y)Z$. For this calculate first

$$\begin{aligned} \nabla_X\nabla_Y(fZ) &= \nabla_X\left((\mathcal{L}_Y f)Z + f\nabla_Y Z\right) \\ &= \left(\mathcal{L}_X\mathcal{L}_Y f\right)Z + (\mathcal{L}_Y f)\nabla_X Z + (\mathcal{L}_X f)\nabla_Y Z + f\nabla_X\nabla_Y Z \end{aligned}$$

This yields

$$\nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) = \left(\mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f\right)Z + f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z$$

and hence

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) + \nabla_{[X,Y]}(fZ) \\ &= \left(\mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f\right)Z + f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z + (\mathcal{L}_{[X,Y]}f)Z + f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z + \Big(\mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f + \mathcal{L}_{[X,Y]}f\Big)Z \\ &= fR(X,Y)Z \end{aligned}$$

The last step uses Exercise 2 in Exercise Sheet 4, where we proved $\mathcal{L}_Y\mathcal{L}_X f - \mathcal{L}_X\mathcal{L}_Y f = \mathcal{L}_{[X,Y]}f$.

b) We will make extensive use of part a) in the following calculations. We show first $(\nabla_{fX}R)(Y,Z)W = f(\nabla_X R)(Y,Z)W$:

$$\begin{aligned}
&(\nabla_{fX}R)(Y,Z)W\\
&\quad= \nabla_{fX}(R(Y,Z)W) - R(\nabla_{fX}Y,Z)W - R(Y,\nabla_{fX}Z) - R(Y,Z)(\nabla_{fX}W)\\
&\quad= f\nabla_X(R(Y,Z)W) - R(f\nabla_X Y,Z)W - R(Y,f\nabla_X Z) - R(Y,Z)(f\nabla_X W)\\
&\quad= f\Big(\nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z) - R(Y,Z)(\nabla_X W)\Big)\\
&\quad= f(\nabla_X R)(Y,Z)W
\end{aligned}$$

Thereafter, we show $(\nabla_X R)(fY,Z)W = f(\nabla_X R)(Y,Z)W$:

$$\begin{aligned}
&(\nabla_X R)(fY,Z)W\\
&\quad= \nabla_X(R(fY,Z)W) - R(\nabla_X(fY),Z)W - R(fY,\nabla_X Z)W - R(fY,Z)(\nabla_X W)\\
&\quad= \nabla_X(fR(Y,Z)W) - R((\mathcal{L}_X f)Y + f\nabla_X Y,Z)W - fR(Y,\nabla_X Z) - fR(Y,Z)(\nabla_X W)\\
&\quad= (\mathcal{L}_X f)R(Y,Z)W + f\nabla_X\left(R(Y,Z)W\right) - (\mathcal{L}_X f)R(Y,Z)W - fR(\nabla_X Y,Z)W\\
&\qquad - fR(Y,\nabla_X Z) + fR(Y,Z)(\nabla_X W)\\
&\quad= f(\nabla_X R)(Y,Z)W
\end{aligned}$$

By anti-symmetry, this also yields the third equation:

$$(\nabla_X R)(Y,fZ)W = -(\nabla_X R)(fZ,Y)W = -f(\nabla_X R)(Z,Y)W = f(\nabla_X R)(Z,Y)W$$

Finally, we verify $(\nabla_X R)(Y,Z)(fW) = f(\nabla_X R)(Y,Z)W$:

$$\begin{aligned}
&(\nabla_X R)(Y,Z)(fW)\\
&\quad= \nabla_X(R(Y,Z)(fW)) - R(\nabla_X(Y),Z)(fW) - R(Y,\nabla_X Z)(fW) - R(Y,Z)(\nabla_X(fW))\\
&\quad= \nabla_X(fR(Y,Z)W) - fR(\nabla_X Y,Z)W - fR(Y,\nabla_X Z)W + R(Y,Z)((\mathcal{L}_X f)W - f\nabla_X W)\\
&\quad= (\mathcal{L}_X f)R(Y,Z)W + f\nabla_X\left(R(Y,Z)W\right) - fR(Y,Z)W - fR(Y,\nabla_X Z)W\\
&\qquad - (\mathcal{L}_X f)R(Y,Z)W + fR(Y,Z)(\nabla_X W)\\
&\quad= f(\nabla_X R)(Y,Z)W
\end{aligned}$$

c) The proof is very general and applies to all kind of tensors. We proceed in two steps.

Step 1: Let $p \in U \subset M$ be an open neighbourhood. Suppose X,Y,Z,W and $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}$ are vector fields on M which agree on U respectively. Then $\nabla_X R(Y,Z)W(p) = \nabla_{\tilde{X}} R(\tilde{Y},\tilde{Z})\tilde{W}(p)$.

Let $\rho : M \to \mathbb{R}$ be a smooth function satisfying $\rho(p) = 0$ and $\rho|_{M\setminus U} \equiv 1$. Then follows $(X - \tilde{X}) = \rho(X - \tilde{X})$ and by part b)

$$\begin{aligned}
\nabla_X R(Y,Z)W(p) - \nabla_{\tilde{X}} R(Y,Z)W(p) &= \nabla_{\rho(X-\tilde{X})} R(Y,Z)W(p)\\
&= \rho(p)\nabla_{(X-\tilde{X})} R(Y,Z)W(p) = 0.
\end{aligned}$$

This shows $\nabla_X R(Y,Z)W(p) = \nabla_{\tilde{X}} R(Y,Z)W(p)$. Repeating the same argument we obtain

$$\nabla_{\tilde{X}} R(Y,Z)W(p) - \nabla_{\tilde{X}} R(\tilde{Y},Z)W(p) = \rho(p)\nabla_{\tilde{X}} R(Y-\tilde{Y},Z)W = 0$$

$$\nabla_{\tilde{X}} R(\tilde{Y}, Z)W(p) - \nabla_{\tilde{X}} R(\tilde{Y}, \tilde{Z})W(p) = \rho(p)\nabla_{\tilde{X}} R(\tilde{Y}, Z - \tilde{Z})W = 0$$
$$\nabla_{\tilde{X}} R(\tilde{Y}, \tilde{Z})W(p) - \nabla_{\tilde{X}} R(\tilde{Y}, \tilde{Z})\tilde{W}(p) = \rho(p)\nabla_{\tilde{X}} R(\tilde{Y}, \tilde{Z})(W - \tilde{W}) = 0$$

and this establishes Step 1.

Step 2: $\nabla_X R(Y, Z)W(p)$ depends only on $X(p), Y(p), Z(p), W(p) \in T_pM$.

Let $E_1, \dots E_m \in \mathrm{Vect}(M)$ be vector fields such that $E_1(p), \dots, E_m(p)$ forms a basis of T_pM. As linear independence is an open condition, there exists an open neighbourhood $p \in U \subset M$ such that $E_1(x), \dots, E_m(x)$ forms a basis of T_xM for every $x \in U$. In particular, there exists smooth functions $X^i, Y^j, Z^k, W^\ell \in C^\infty(M, \mathbb{R})$ defining vector fields

$$\tilde{X}(x) = \sum_{i=1}^m X^i(x)E_i(x), \qquad \tilde{Y}(x) = \sum_{j=1}^m Y^j(x)E_j(x),$$
$$\tilde{Z}(x) = \sum_{k=1}^m Z^k(x)E_k(x), \qquad \tilde{W}(x) = \sum_{\ell=1}^m W^\ell(x)E_\ell(x)$$

such that X, Y, Z, W and $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ agree on U. It follows now from Step 1 and part b):

$$\begin{aligned}\nabla_X R(Y, Z)W(p) &= \nabla_{\tilde{X}} R(\tilde{Y}, \tilde{Z})\tilde{W}(p) \\ &= \sum_{i,j,k,\ell=1}^m X^i(p)Y^j(p)Z^k(p)W^\ell(p)\nabla_{E_i} R(E_j, E_k)E_\ell\end{aligned}$$

The last expression depends only on $X(p), Y(p), Z(p), W(p) \in T_pM$ and this proves the claim.

2. Let $E_i = \frac{\partial \psi}{\partial x^i}$ for $i = 1, \dots, m$ and for $\psi : \Omega \to M$ a parametrisation. Take $\nabla_i R^\nu_{jk\ell}$ to be given by $\sum_{\nu=1}^m \nabla_i R^\nu_{jk\ell} E_\nu = (\nabla_{E_i} R)(E_j, E_k)E_\ell$. Prove that

$$\nabla_i R^\nu_{jk\ell} = \partial_i R^\nu_{jk\ell} + \sum_{\mu=1}^m \Gamma^\nu_{i\mu} R^\mu_{jk\ell} - \sum_{\mu=1}^m \left(\Gamma^\mu_{ij} R^\nu_{\mu k\ell} + \Gamma^\mu_{ik} R^\nu_{j\mu\ell} + \Gamma^\mu_{i\ell} R^\nu_{jk\mu}\right).$$

Solution: We use Einstein summation convention, whereby repeated indices are summed from 1 to m. These suppressed summation signs turn up all over the literature especially if you go towards General Relativity and the related field of Geometric Analysis.

We start calculating

$$\begin{aligned}\nabla_i R^\nu_{jk\ell} E_\nu &= (\nabla_{E_i} R)(E_j, E_k)E_\ell \\ &= \nabla_{E_i}(R^\mu_{jk\ell} E_\mu) - R(\nabla_{E_i} E_j, E_k)E_\ell - R(E_j, \nabla_{E_i} E_k)E_\ell - R(E_j, E_k)\nabla_{E_i} E_\ell \\ &= \left(\partial_i R^\nu_{jk\ell} + R^\mu_{jk\ell}\Gamma^\nu_{i\mu} - R^\nu_{\mu k\ell}\Gamma^\mu_{ij} - R^\nu_{j\mu\ell}\Gamma^\mu_{ik} - R^\nu_{jk\mu}\Gamma^\mu_{i\ell}\right) E_\nu,\end{aligned}$$

where we used the Leibniz rule and the relation $\nabla_{E_i} E_j = \Gamma^\mu_{ij} E_\mu$.

3. Prove the second Bianchi identity, i.e. prove that

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0.$$

Solution: The covariant derivative $(\nabla_X R)(Y,Z)$ is defined by

$$(\nabla_X R)(Y,Z)W = \nabla_X\left(R(Y,Z)W\right) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W - R(X,Y)\nabla_X W.$$

We can write the cyclic sum as

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = A - B + C$$

where

$$\begin{aligned} A &= \nabla_X(R(Y,Z)W) + \nabla_Y(R(Z,X)W) + \nabla_Z(R(X,Y)W), \\ B &= R(Y,Z)\nabla_X W + R(Z,X)\nabla_Y W + R(X,Y)\nabla_Z W, \\ C &= R([X,Y],Z)W + R([Y,Z],X)W + R([Z,X],Y)W. \end{aligned}$$

where for C we used anti-symmetry of the curvature tensor ($R(X,Y) = -R(Y,X)$) and the torsion-free identity $[X,Y] = \nabla_Y X - \nabla_X Y$.

Plug in the definition of the curvature tensor in the expression A and B. All triple covariant derivatives cancel out and the remaining terms are the following.

$$\begin{aligned} A - B =& \nabla_X\nabla_{[Y,Z]}W - \nabla_{[Y,Z]}\nabla_X W + \nabla_Y\nabla_{[Z,X]}W - \nabla_{[Z,X]}\nabla_Y W \\ &+ \nabla_Z\nabla_{[X,Y]}W - \nabla_{[X,Y]}\nabla_Z W. \end{aligned}$$

Now plug in the definition of the curvature tensor in C. The double covariant derivatives in C cancel with the double covariant derivatives in $A - B$ and we get

$$\begin{aligned} A - B + C &= \nabla_{[[X,Y],Z]}W + \nabla_{[[Y,Z],X]}W + \nabla_{[[Z,X],Y]}W \\ &= \nabla_{[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]}W = 0. \end{aligned}$$

The last equality follows from the Jacobi-identity.

4. Let M be a connected symmetric manifold.

a) Prove that M is complete.

b) For $p \in M$ denote by $\phi_p : M \to M$ the isometry with $\phi_p(p) = p$ and $d\phi_p(p) = -\mathbb{1}$. Prove that the map

$$\sigma : M \times M \to M, \qquad (p,q) \mapsto \phi_p(q)$$

is smooth.

c) Let γ be a non-constant geodesic. Prove that the map $\tau : (\mathbb{R},+) \to (\mathrm{Isom}(M),\circ) : t \mapsto \tau_{\gamma,t} := \phi_{\gamma(t/2)} \circ \phi_{\gamma(0)}$ is a group homomorphism.

d) Prove that M is homogeneous.

Hint: Use Hopf-Rinow for b) and d), which is possible by a). In b), express $\phi_w(q)$ in terms of exponential maps and parallel transport for w in a geodesically convex neighbourhood of p. For c), see that τ is translation along the geodesic γ and use uniqueness of isometries. Additionally, the images of parallel vector fields come in handy in b) and c).

Solution:

a) Let $\gamma : I = (a,b) \to M$ be a geodesic where I is the maximal existence interval and assume $b < \infty$. Choose $\frac{a+b}{2} < t_0 < b$, put $p := \gamma(t_0)$. Then $\gamma' := \phi_p(\gamma(t_0 - (t - t_0)))$ extend γ beyond b. This is a contradiction and thus M is complete.

b) The map $\sigma : M \times M \to M$ is smooth if and only if it is smooth in each variable. The map $\sigma(p, \cdot) = \phi_p : M \to M$ is a smooth isometry by definition. We show in the following that for fixed $q \in M$ the map $\phi(\cdot, q) : M \to M$, $p \mapsto \phi_p(q)$ is smooth.

Let $p, q \in M$ be given and choose $\mathrm{inj}(p) > r > 0$ such that $B_r(p)$ is geodesically convex. It suffices to show that the map

$$\Phi : B_{r/3}(p) \to M, \qquad w \mapsto \phi_w(q)$$

is smooth. By what we prove in a), we know that M is complete. Then by Hopf-Rinow, there is $\eta \in T_pM$ such that $\exp_p(\eta) = q$. Moreover, for every $w \in B_{r/3}(p)$ there is a unique $\nu = \nu(w) \in T_pM$ such that $\exp_p(\nu) = w$ and $d(p, w) = |\nu|$. This is given by the inverse of the exponential map

$$\nu(w) := \left(\exp_p|_{B_{r/3}(0;T_pM)}\right)^{-1}(w)$$

and so $\nu(w)$ depends smoothly on w. Since ϕ_w is an isometry, we have

$$\Phi(w) = \phi_w(q) = \phi_w(\exp_p(\eta)) = \exp_{\phi_w(p)}(d\phi_w(p)\eta).$$

Since ϕ_w is the reflection in w, we get $\phi_w(p) = \exp_p(2\nu(w))$ by geodesic convexity. In particular, the map $w \mapsto \phi_w(p)$ is smooth.

Now define a vector field $X \in \mathrm{Vect}(B_r(p))$ satisfying $X(p) = \eta$ and for every $\hat{p} \in T_pM$ with $|\hat{p}| < r$

$$\nabla X(\gamma_{\hat{p}}(t)) = 0$$

is parallel along the curve $\gamma_{\hat{p}} : [0, 1] \to M$ defined by $\gamma_{\hat{p}}(t) := \exp_p(t\hat{p})$. It follows from smooth dependence of solutions of ODE's on initial conditions that such a vector field X exists and that X is smooth. Since ϕ_w is an isometry, it preserves parallel transport in the following sense: If X is a parallel vector field along γ, then $(\phi_w)_*X$ is a parallel vector field along $\phi_w \circ \gamma$. In particular, consider the parallel vector field $X'(t) = X(\gamma_{2\nu(w)}(t))$ running along $\gamma_{2\nu(w)}$. Denote the push-forward vector field by $X''(t)$. This is a vector field along the curve $\phi_w \circ \gamma = \gamma_{2\nu(w)}(2 - t)$ and satisfies $-X''(1/2) = X'(1/2) = X(w) \in T_wM$ (since $d\phi_w(w) = -\mathbb{1}$). Since $X''(t)$ is parallel, we deduce that $X''(t) = -X(1 - t)$ and in particular

$$d\phi_w(p)\eta = X''(0) = -X'(1) = -X(\exp_p(2\nu(w))).$$

In particular, $w \mapsto d\phi_w(p)\eta$ is a smooth function $B_{r/3}(p) \to TM$.

So in summary, we have shown that

$$\Phi(w) = \phi_w(q) = \exp_{\phi_w(p)}(d\phi_w(p)\nu) = \exp_{\exp_p(2\nu(w))}(-X(\exp_p(2\nu(w)))).$$

is a smooth function and this completes the proof.

c) Note that

$$\tau : \mathbb{R} \to \mathrm{Isom}(M)$$

is well defined since $\tau_{\gamma,t}$ is the composition of isometries. On the other hand it is easy see

$$\tau_{\gamma,s}\gamma(t) = \phi_{\gamma(s/2)}\phi_{\gamma(0)}(\gamma(t)) = \phi_{\gamma(s/2)}\gamma(-t) = \gamma(s + t).$$

and so $\tau_{\gamma,s} \circ \tau_{\gamma,t}(\gamma(0)) = \tau_{\gamma,s+t}(\gamma(0))$. Furthermore, if X is a parallel vector field along γ, then under $\phi_{\gamma(0)}$ it gets mapped to the parallel vector field X' along $\gamma' : t \mapsto \gamma(-t)$ having initial condition $X'(0) = -X(0)$ i.e. $X'(t) = -X(-t)$. Then applying $\phi_{\gamma(s/2)}$ to X', we get a parallel vector field along $\gamma'' : t \mapsto \gamma(s+t)$ such that $X''(-s/2) = -X'(-s/2) = X(s/2)$, i.e. $X''(t) = X(t+s)$. Thus

$$d\tau_{\gamma,s}(\gamma(0))X(0) = X(s)$$

and thus

$$d\left(\tau_{\gamma,t+s}\right)_{\gamma(0)} = d\left(\tau_{\gamma,s} \circ \tau_{\gamma,t}\right)_{\gamma(0)}.$$

Hence the two isometries are the same, i.e

$$\tau_{\gamma,t+s} = \tau_{\gamma,s} \circ \tau_{\gamma,t}.$$

by uniqueness of isometry.

d) By Hopf-Rinow there exists a minimising geodesic $\gamma : [0,\ell] \to M$ between any two pair $p, q \in M$. Let $p' := \gamma(\frac{\ell}{2})$. Then

$$\phi_{p'}(p) = \phi_{p'}(\gamma(0)) = q$$

and so Isom(M) acts transitively on M.

5. **a)** Show that S^n is a symmetric space. For this verify that the map $\phi_p : S^n \to S^n$ given by $\phi_p(x) = -x + 2\langle p, x\rangle p$ is an isometry satisfying $\phi_p(p) = p$ and $d\phi_p(p) = -\mathbb{1}$.

b) Show that a compact Lie subgroup $G \subset O(n)$ is a symmetric space. For this verify that the map $\phi_a : G \to G$ given by $\phi_a(g) = ag^{-1}a$ is an isometry satisfying $\phi_a(a) = a$ and $d\phi_a(a) = -\mathbb{1}$.

c) Let G be as in part b) above. Show that its sectional curvature is given by

$$K(\mathbb{1}, E) = \frac{1}{4}|[\xi,\eta]|^2$$

where E is a two dimensional subspace of $\mathfrak{g} = T_{\mathbb{1}}G$ and ξ, η an orthonormal basis for E.

Hint: For b) and c) recall exercise 2 of sheet 10 and exercise 6 of sheet 8. In c) the identity $\langle [\xi,\eta], \zeta\rangle = \langle \xi, [\eta,\zeta]$ for $\xi, \eta, \zeta \in T_{\mathbb{1}}G$ might be handy.

Solution:

a) We have for $x \in S^n$ that $|\phi_p(x)|^2 = |x|^2 - 2\langle x, \langle p, x\rangle p\rangle + 4\langle p, x\rangle^2 = 1$, so ϕ_p is a well-defined map. We also have that $\phi_p^2(x) = x - 2\langle p, x\rangle p - 2\langle p, x\rangle + 4\langle p, x\rangle |p|^2 = x$. So ϕ_p is a diffeomorphism. Furthermore, for $v, w \in T_xS^n$, we get

$$\begin{aligned}\langle d\phi_p(x)v, d\phi_p(x)w\rangle &= \langle -v + 2\langle p, v\rangle p, -w + 2\langle p, w\rangle p\rangle \\ &= \langle v, w\rangle - 4\langle v, w\rangle\langle v, w\rangle + 4\langle v, w\rangle\langle v, w\rangle = \langle v, w\rangle.\end{aligned}$$

So ϕ_p is an isometry. Also $\phi_p(p) = p$ and $d\phi(p) = -\mathbb{1}$. As $p \in S^n$ was arbitrary, S^n is a symmetric space.

Alternatively, one can conclude from the last exercise that S^n has constant sectional curvature, which also implies that S^n is symmetric.

b) One clearly has $\phi_a(a) = aa^{-1}a = a$. We calculate its derivative using the chainrule and obtain

$$d\phi_a(g)\hat{g} = -ag^{-1}\hat{g}g^{-1}a.$$

At $g = a$ this yields $d\phi_a(a)\hat{g} = -\hat{g}$.

It remains to show that ϕ_a is an isometry. Since G is a Lie group, the inversion $I(g) := g^{-1}$ and the translation maps $L_a(g) := ag$, $L_{a^{-1}}(g) := a^{-1}g$ are all diffeomorphism of G. Hence

$$\phi_a(g) = ag^{-1}a = a(a^{-1}g)^{-1} = (L_a \circ I \circ L_{a^{-1}})(g)$$

shows that ϕ_a is a diffeomorphism of G. Next, recall that the euclidean inner product on $\mathbb{R}^{n\times n}$ is given by $\langle \xi, \eta\rangle = \mathrm{tr}(\xi^\top \eta)$. With this we verify:

$$\begin{aligned}
\langle d\phi_a(g)\hat{g}_1, d\phi_a(g)\hat{g}_2\rangle &= \mathrm{tr}\left((ag^{-1}\hat{g}_1 g^{-1}a)^\top (ag^{-1}\hat{g}_2 g^{-1}a)\right) \\
&= \mathrm{tr}\left((g^{-1}a)^\top \hat{g}_1^\top (ag^{-1})^\top (ag^{-1})\hat{g}_2 (g^{-1}a)\right) \\
&= \mathrm{tr}\left((g^{-1}a)^{-1} \hat{g}_1^\top \hat{g}_2 (g^{-1}a)\right) \\
&= \mathrm{tr}\left(\hat{g}_1^\top \hat{g}_2\right) \\
&= \langle \hat{g}_1, \hat{g}_2\rangle
\end{aligned}$$

Hence ϕ_a perserves the first fundamental form of G and thus ϕ_a is an isometry.

c) We show first that

$$\langle [\xi, \eta], \zeta\rangle = \langle \xi, [\eta, \zeta]\rangle, \quad \text{for all } \xi, \eta, \zeta \in T_{\mathbb{1}}G.$$

This follows abstractly from differentiating the identity

$$\langle \eta, \zeta\rangle = \langle g\eta g^{-1}, g\zeta g^{-1}\rangle$$

at $g = \mathbb{1}$ in direction ξ. Alternatively, a direct calculation reveals

$$\langle [\xi, \eta], \zeta\rangle = \mathrm{tr}\left(\zeta^\top(\xi\eta - \eta\xi)\right) = \mathrm{tr}(\zeta\eta\xi) - \mathrm{tr}(\zeta\xi\eta)$$

$$\langle \xi, [\eta, \zeta]\rangle = \mathrm{tr}\left(\xi^\top(\eta\zeta - \zeta\eta)\right) = \mathrm{tr}(\xi\zeta\eta) - \mathrm{tr}(\xi\eta\zeta)$$

where we used that $\xi, \eta, \zeta \in T_{\mathbb{1}}G \subset T_{\mathbb{1}}O(n)$ are skew-symmetric matrices. Both expressions on the right hand side agree, and this establishes again the formula.

Next, recall from Exercise 2 of Exercise Sheet 10 that the curvature tensor of G at $g = \mathbb{1}$ is given by

$$R_{\mathbb{1}}(\xi, \eta)\zeta = -\frac{1}{4}[[\xi, \eta], \zeta].$$

for $\xi, \eta, \zeta \in T_{\mathbb{1}}G$. Now suppose ξ, η is an orthonormals basis of $E \subset T_{\mathbb{1}}G$. Then

$$\begin{aligned}
K(\mathbb{1}, E) &= \langle R_{\mathbb{1}}(\xi, \eta)\eta, \xi\rangle \\
&= -\frac{1}{4}\langle [[\xi, \eta], \eta], \xi\rangle \\
&= -\frac{1}{4}\langle [\xi, \eta], [\eta, \xi]\rangle \\
&= \frac{1}{4}|[\xi, \eta]|^2.
\end{aligned}$$

6. Prove that the sphere $S^n_r; = \{x \in \mathbb{R}^{n+1} : |x| = r\}$ of radius $r > 0$ has sectional curvature $K(p, E) = \frac{1}{r^2}$.

Solution: We have for a hypersurface M in $\mathbb{R}^{n+1}$ with unit normal ν, the orthogonal projection onto T_pM is given by

$$\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^\top$$

and so

$$d\Pi(p)u = -\nu(p)(d\nu(p)u)^\top - (d\nu(p)u)\nu(p)^\top.$$

So

$$h_p(v) = \nu(p)(d\nu(p)v)^\top : T_pM \to T_pM^\perp, \quad h_p(u)^* = (d\nu(p)u)\nu(p)^\top : T_pM^\perp \to T_pM.$$

Therefore, Gauß-Codazzi formula gives

$$R_p(u, v)w = h_p(u)^*h_p(v)w - h_p(v)^*h_p(u)w = \langle d\nu(p)v, w\rangle d\nu(p)u - \langle d\nu(p)u, w\rangle d\nu(p)v.$$

So we obtain for an orthonormal basis $u, v \in T_pM$ of a subspace E that

$$K(p, E) = \langle d\nu(p)u, u\rangle\langle d\nu(p)v, v\rangle - \langle d\nu(p)u, v\rangle\langle d\nu(p)v, u\rangle.$$

For S^n_r, we have that $\nu(p) = p/r$. Thus $d\nu(p)\hat{p} = \hat{p}/r$ and

$$K(p, E) = \left\langle \frac{u}{r}, u \right\rangle \left\langle \frac{v}{r}, v \right\rangle - \left\langle \frac{u}{r}, v \right\rangle \left\langle \frac{v}{r}, u \right\rangle = \frac{1}{r^2}.$$