ETH Zurich HS 2017
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{A}, P) = \Big((0,1), \mathcal{R}, \mu\Big)$, where μ is the Lebesgue measure over $(0,1)$ and $\mathcal{R}$ the Borel σ-algebra on $(0,1)$. Find the distribution function of the random variable

$$X(\omega) := \frac{1}{\lambda} \log \frac{1}{1-\omega}$$

where λ is a given positive parameter.

Exercise 1.2 Let $\mathcal{Z} := (A_i)_{i \in I}$ be a countable decomposition of a set $\Omega \neq \emptyset$ in "atoms" A_i, that is $\Omega = \bigcup_{i \in I} A_i$, where $A_i \cap A_k = \emptyset$ for $i \neq k$, and I countable.

(a) Show that the σ-algebra generated by $\mathcal{Z}$ is of the form

$$\sigma(\mathcal{Z}) = \left\{ \bigcup_{i \in J} A_i \,\middle|\, J \subseteq I \right\}.$$

Hint: Recall the definition of $\sigma(\mathcal{Z})$.

(b) Show that the family of $\sigma(\mathcal{Z})$-measurable random variables is exactly the family of functions on Ω that are constant on "atoms" (that is, all functions f such that for each i, f is constant on A_i).

Exercise 1.3 Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(A_n)_{n \in \mathbb{N}}$ a sequence of sets from $\mathcal{A}$. We define

$$\bar{A} := \limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k \quad , \quad \underline{A} := \liminf_{n \to \infty} A_n := \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k.$$

Let 1_B denote the indicator function of $B \in \mathcal{A}$.

(a) Show that $1_{\bar{A}} = \limsup_{n \to \infty} 1_{A_n}$ and that $1_{\underline{A}} = \liminf_{n \to \infty} 1_{A_n}$.

(b) Show that $P[\underline{A}] \leq \liminf_{n \to \infty} P[A_n]$ and that $P[\bar{A}] \geq \limsup_{n \to \infty} P[A_n]$.
Hint: Use a lemma from Section 1.2 in the lecture notes.

Exercise 1.4 Let Ω be a non-empty set and let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two functions. The σ-algebra on Ω generated by X is defined by $\sigma(X) := \Big\{ X^{-1}(B) \mid B \in \mathcal{R} \Big\}$,

where $\mathcal{R}$ denotes the Borel σ-algebra on $\mathbb{R}$. In this exercise we will show that:
Claim: Y is $\sigma(X)$-$\mathcal{R}$-measurable $\iff$ there exists an $\mathcal{R}$-$\mathcal{R}$-measurable function $f : \mathbb{R} \to \mathbb{R}$, such that $Y = f \circ X$.
Hint: For (b)–(e), cf. the proof of (1.2.16) in the lecture notes.

(a) Show the $\Longleftarrow$ direction.

(b) Show the $\Longrightarrow$ direction for any Y of the form $Y = 1_A$, where $A \in \sigma(X)$.

(c) Show the $\Longrightarrow$ direction for any Y that is a linear combination of indicator functions, i.e. for Y of the form $Y = \sum_{i=1}^{n} c_i 1_{A_i}$, where $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \sigma(X)$.

(d) Show the $\Longrightarrow$ direction for any Y such that $Y \geq 0$.

(e) Complete the proof of the claim (i.e. show the $\Longrightarrow$ direction for an arbitrary Y).

Submission deadline: 13:15, Oct 3.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon