ETH Zurich HS 2017
D-MATH, D-PHYS
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Coordinator
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Probability Theory

Exercise Sheet 3

Exercise 3.1 Let $\mathcal{M}$ be the set of the real-valued random variables on the probability space $(\Omega, \mathcal{A}, P)$. We define on $\mathcal{M}$ an equivalence relation as follows:

$$X \sim Y \quad :\Longleftrightarrow \quad P(X = Y) = 1$$

We denote by $\mathcal{M}/\sim$ the set of equivalence classes in $\mathcal{M}$ with respect to $\sim$ and we denote by $[X]$ the equivalence class of $X \in \mathcal{M}$.

(a) Show that

$$\begin{aligned} d : (\mathcal{M}/\sim) \times (\mathcal{M}/\sim) &\rightarrow \mathbb{R} \\ ([X],[Y]) &\mapsto E\left[|X - Y| \wedge 1\right] \end{aligned}$$

is a metric on $\mathcal{M}/\sim$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and let X be an element of $\mathcal{M}$. Show that $([X_n])_{n\in\mathbb{N}}$ converges to $[X]$ with respect to the metric d if and only if $(X_n)_{n\in\mathbb{N}}$ converges to X in probability.

Exercise 3.2 Let X_i, $i \geq 1$, be identically distributed, integrable random variables and define $S_n = \sum_{i=1}^n X_i$ for each $n \in \mathbb{N}$. Show that:

$$\lim_{M\to\infty} \sup_{n\geq 1} E\left[\frac{|S_n|}{n} 1_{\left\{\frac{|S_n|}{n} > M\right\}}\right] = 0.$$

Note: This family $\left\{\frac{|S_n|}{n}, n \in \mathbb{N}\right\}$ is thus so-called "uniformly integrable". See (3.6.14) in the lecture notes. Thanks to Theorem 3.41 and the strong law of large numbers, one has that: if $X_i, i \geq 1$, are also pairwise independent, (in addition to being identically distributed as in the question), then $\frac{S_n}{n}$ converges P-a.s. and in L^1 towards $E[X_1]$ for $n \to \infty$.

Exercise 3.3 Let $X_n, n \geq 0$, be i.i.d. $\mathcal{N}(0,1)$ random variables and let $t \in [0,1]$.

(a) Show that

$$tX_0 + \sum_{n=1}^{\infty} \sqrt{2}\, \frac{\sin(n\pi t)}{n\pi}\, X_n$$

converges in L^2 and P-a.s. *Hint:* Use Theorem 1.34 and Remark 1.35 of lecture notes.

(b) Denote by B_t the limit of the series above. Show that for $0 \leq t, s \leq 1$,

$$E[B_t] = 0 \quad \text{and} \quad E[B_t B_s] = t \wedge s,$$

where $t \wedge s = \min(t, s)$. *Hint:* Find an orthonormal basis of $L^2([0,1], dx)$ (e.g. trigonometric functions) and use Parseval-Bessel identity for a Hilbert space (see, for example, Theorem 4.18 in *Real and Complex Analysis by W. Rudin*).

Submission deadline: 13:15, Oct 17.

Location: During exercise class or in the tray outside of HG E 65.

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Class assignment:

Students	Time & Date	Room	Assistant
An-Gr	Tue 13-14	HG F 26.5	Yilin Wang
He-Lang	Tue 13-14	ML H 41.1	Angelo Abächerli
Lanz-Sa	Tue 14-15	HG F 26.5	Vincenzo Ignazio
Sch-Zh	Tue 14-15	ML H 41.1	Lukas Gonon